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White Noise Approach to Multiparameter Stochastic Integration.

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WHITE NOISE APPROACH TO MULTIPARAMETER STOCHASTIC INTEGRATION

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TABLE OF CONTENTS

ABSTRACT

In this dissertation we will set up the Hida theory of generalized Brownian functionals, or white noise analysis, on $\mathscr{S}^*(\mathbb{R}^d)$, the space of tempered distributions, and apply it to multiparameter stochastic integration. With the partial ordering on $\mathbb{R}^d_+$: $(s_1,\ldots,s_d) < (t_1,\ldots,t_d)$ if $s_i < t_i$, $1 \leq i \leq d$, the Wiener process

$$W((t_1,\ldots,t_d),x) = \langle x, 1_{[0,t_1)\times\cdots\times[0,t_d)}\rangle, \; x \in \mathscr{S}^*(\mathbb{R}^d)$$

is a generalization of a Brownian motion and there is the Wiener – Ito decomposition: $L^2(\mathscr{S}^*(\mathbb{R}^d)) = \sum_{n=0}^{\infty} \oplus K_n$, where K_n is the space of n–tuple Wiener integrals. As in the one–dimensional case, there are the continuous inclusions

$$(L^2)^+ \subset L^2(\mathscr{S}^*(\mathbb{R}^d)) \subset (L^2)^-,$$

and $(L^2)^-$ is considered the space of generalized Wiener functionals. We will define the differentiation operator $\partial_{(t_1,\ldots,t_d)}$ and its adjoint $\partial^*_{(t_1,\ldots,t_d)}$ and give some properties. We prove that the multidimensional time Ito stochastic integral is a special case of a white noise integral and give conditions for its existence. For $d = 2$ the Ito integral is not sufficient for representing elements of $L^2(\mathscr{S}^*(\mathbb{R}^2))$. We show that the other integral involved can also be realized in the white noise setting. For $F \in \mathscr{S}^*(\mathbb{R}^d)$ we will then define $F(W((s,t),x)$ as an element of $(L^2)^-$ and obtain a generalized Ito formula.

Introduction

The Ito theory of stochastic differential equations provides a probabilistic method for generating diffusion processes. The theory rests on the definition of the Ito stochastic integral:

$$\int_{t_0}^{t} \varphi(s,\omega)dB(s,\omega).$$

This is a random variable defined on a probability space on which there is a Brownian motion $\{B(t,\omega);\ t \geq 0\}$: a Gaussian system for which (a) $B(0,\omega) = 0$ with probability 1, (b) for almost all ω, $B(\cdot,\omega)$ is continuous, and (c) for each t, B(t) is normally distributed with mean 0 and $E(B(t)B(s)) = t \wedge s$. Denote by $\mathscr{F}_t$ the sigma field generated by the random variables $\{B(s,\cdot);\ t_0 \leq s \leq t\}$. For the Ito integral to exist, $\varphi(\cdot,\omega)$ must be in $L^2[t_0,t]$ with probability 1 and φ must be non–anticipating, i.e., for each $s \in [t_0,t]$, $\varphi(s,\cdot)$ must be measurable with respect to $\mathscr{F}_s$. If for a partition $\{t_i\}_{i=1}^n$ of $[t_0,t]$, $\varphi(s,\omega) = \sum_{i=1}^{n} \alpha_i(\omega)1_{[t_{i-1},t_i[}(s)$ almost everywhere, then

$$\int_{t_0}^{t} \varphi(s,\omega)dB(s,\omega) := \sum_{i=1}^{n} \alpha_i(\omega)(B(t_i,\omega) - B(t_{i-1},\omega)) = \sum_{i=1}^{n} \alpha_i(\omega)\Delta_i B.$$

A process φ satisfying the existence conditions can be approximated with such simple functions and the corresponding sequence of stochastic integrals converges in probability. The limit is then defined to be $\int_{t_0}^{t} \varphi(s,\omega)dB(s,\omega)$. Diffusion processes are obtained as solutions of integral equations of the form

$$X_t = c + \int_{t_0}^{t} f(s,X_s)ds + \int_{t_0}^{t} G(s,X_s)dB(s).$$

The fundamental result of this theory is the Ito formula, the simplest version of which is:

$$F(B(t)) - F(B(0)) = \int_0^t F'(B(s))dB(s) + \tfrac{1}{2}\int_0^t F''(B(s))ds.$$

In the early seventies Cairoli [1], Cairoli and Walsh [2], and Wong and Zakai [20] began the development of stochastic integration for processes with multidimensional parameter. Such processes arise in quantum field theory and "computer image processing". They defined an Ito–type integral

$$\int_{[a,b]^d} \varphi(\mathbf{x},\omega)dW(\mathbf{x},\omega).$$

Here $\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}_+^d = \{(x_1,\ldots,x_d);\ x_i \geq 0,\ 1 \leq i \leq d\}$ and $W(\mathbf{x},\omega)$ is a Wiener process: $W(\mathbf{0},\omega) = 0$ almost everywhere, $\{W(\mathbf{x},\omega);\ \mathbf{x} \in \mathbb{R}_+^d\}$ is Gaussian, $W(\mathbf{x},\omega)$ has mean 0 with $E(W(\mathbf{x})W(\mathbf{y})) = \prod_{i=1}^{d} (x_i \wedge y_i)$, and for almost all ω, $W(\cdot,\omega)$ is continuous. The construction is similar to the one–dimensional case. Using the partial ordering $(x_1,\ldots,x_d) < (y_1,\ldots,y_d)$ if $x_i \leq y_i$ for $1 \leq i \leq d$, the measurability requirement is with respect to the sigma fields $\mathcal{F}_{\mathbf{x}}$ generated by $\{W(\mathbf{y},\omega);\ \mathbf{y} < \mathbf{x}\}$ and the increment is over rectangles: For $\Delta = \prod_{i=1}^{d} [a_i,b_i[$, $\Delta W(\omega) := \sum_{\mathbf{x}}(-1)^{\pi(\mathbf{x})}W(\mathbf{x},\omega)$, the sum being taken over the 2^d vertices $\{\mathbf{x};\ x_i = a_i \text{ or } b_i\}$ and $\pi(\mathbf{x})$ is the number of b_i's in $\mathbf{x}$. Using martingale theory, Wong and Zakai (also Cairoli and Walsh) obtained an Ito formula for $d = 2$. A second type of integral, the Wong–Zakai integral, was required:

$$\left[\int_{[a,b]^2 \times [a,b]^2}\right] \varphi(\mathbf{z},\mathbf{z}',\omega)dW(\mathbf{z})dW(\mathbf{z}').$$

For this integral to exist $E\int_{[a,b]^2}\int_{[a,b]^2}|\varphi(z,z',\omega)|^2 dz dz'$ must be finite and $\varphi(z,z',\omega)$ must be measurable with respect to $\mathscr{F}_{z\vee z'}$. If $[0,b] = [0,b_1]\times[0,b_2]$ and $F(W(z))$ is a martingale on increasing paths, the 2–dimensional Ito formula is:

$$F(W(b_1,b_2)) - F(W(b_1,0)) - (F(W(0,b_2)) - F(W(0,0))) =$$

$$\int_{[0,b]} F'(W(z))dW(z) + \tfrac{1}{2}\left[\int_{[0,b]\times[0,b]}\right] F''(W(z\vee z'))dW(z)dW(z').$$

Hida [4, 1975] initiated the study of functions of Brownian motion from a white noise point of view and this theory has been applied to stochastic integration. A Brownian motion gives rise to a generalized stochastic process $\{\langle \dot{B},\xi\rangle;\ \xi \in \mathscr{S}\}$ where $\mathscr{S}$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. $\langle \dot{B},\xi\rangle := -\int_{\mathbb{R}} B(t)\xi'(t)dt$ and the probability distribution μ of this process is on $\mathscr{S}^*$, the space of tempered distributions. $(\mathscr{S}^*,\mu)$ is called the standard white noise space. For $t \geq 0$ and $x \in \mathscr{S}^*$, $B(t,x) = \langle x, 1_{[0,t]}\rangle$ is a Brownian motion, and $L^2(\mathscr{S}^*)$ has a direct sum decomposition $\sum_{n=0}^{\infty} \oplus K_n$ where $K_0 = \mathbb{R}$ and K_n is the space of n–tuple Wiener integrals. The elements φ of K_n are given by

$$\varphi = \int_{\mathbb{R}^n} f(u_1,\ldots u_n)dB(u_1)\ldots dB(u_n)$$

where f is a symmetric square integrable function on $\mathbb{R}^n$. By restricting the integrands to a certain Sobolev space one obtains a Hilbert subspace $K_n^{(n)}$ of K_n. The dual $K_n^{(-n)}$ of this subspace is called the space of generalized n–tuple Wiener integrals. Defining $(L^2)^+ = \sum_{n=0}^{\infty} \oplus K_n^{(n)}$ and $(L^2)^- = \sum_{n=0}^{\infty} \oplus K_n^{(-n)}$, we have

$$(L^2)^+ \subset L^2(\mathscr{S}^*) \subset (L^2)^-.$$

White noise calculus is the analysis of the space $(L^2)^-$ thinking of $\{\dot{B}(t);\ t \in \mathbb{R}\}$ as a coordinate system. The coordinate differentiation ∂_t is defined by means of a transformation to a space of functionals on $\mathscr{S}$ and ∂_t: $K_n^{(n)} \to K_{n-1}^{(n-1)}$. The adjoint operator $\overset{*}{\partial}_t$ is defined by duality.

Using white noise calculus Kubo and Takenaka [12, 1981] showed that for a non–anticipating process $\varphi(t,x)$, $x \in \mathscr{S}^*$, such that $E\int_a^b |\varphi(t)|^2 dt < \infty$ the Ito integral can be expressed as a white noise integral:

$$\int_a^b \varphi(t,x)dB(t,x) = \int_a^b \overset{*}{\partial}_t \varphi(t)dt.$$

The integral $\int_a^b \overset{*}{\partial}_t \varphi(t)dt$ can, however, be defined in a more general setting. For a stochastic process $\varphi(t,x)$ on $(\mathscr{S}^*,\mu)$ such that $E\int_a^b |\varphi(t,x)|^2 dt < \infty$, Kuo and Russek [17] proved that $\int_a^b \overset{*}{\partial}_t \varphi(t)dt$ exists. Thus there are no special measurability requirements for this white noise integral. Furthermore, it is an ordinary integral with respect to the time parameter. In 1983, Kubo [11] defined the generalized Brownian functional $F(B(t))$ for $t > s > 0$ and $F \in \mathscr{S}^*$ and developed a generalized Ito formula:

$$F(B(t)) - F(B(s)) = \int_s^t \overset{*}{\partial}_u F'(B(u))du + \tfrac{1}{2}\int_s^t F''(B(u))du.$$

If one considers $\mathscr{S}(\mathbb{R}^d)$, $d > 1$, there is also the d–dimensional white noise space $(\mathscr{S}^*(\mathbb{R}^d),\mu)$ and for $\mathbf{a} = (a_1,\ldots a_d)$ in $\mathbb{R}^d$ with $a_i \geq 0$, the process $W(\mathbf{a},x) = \langle x, 1_{R_{\mathbf{a}}} \rangle$, where $R_{\mathbf{a}} = \prod_{i=1}^{d} [0,a_i[$, is a Wiener process. In this dissertation we will apply the white noise theory and obtain results analogous to those mentioned above for processes with multidimensional time parameter $\mathbf{t} = (t_1,\ldots,t_d)$. We will first set up the Hida theory on $L^2(\mathscr{S}^*(\mathbb{R}^d))$. We will then show that, just as in the 1–dimensional parameter case, the Ito integral for a multidimensional time process is given by a white noise integral:

$$\int_{[a,b]^d} \varphi(t,x)\,dW(t,x) = \int_{[a,b]^d} \overset{*}{\partial}_t \varphi(t,x)\,dt.$$

We will prove that the Wong–Zakai integral, when it exists, can also be represented in the white noise setting:

$$\left[\int_{[a,b]^2\times[a,b]^2}\right] \varphi(z,z')\,dW(z)\,dW(z') = \int_{[a,b]^2}\int_{[a,b]^2} \overset{*}{\partial}_{z'} \overset{*}{\partial}_z 1_G(z,z')\varphi(z,z')\,dz\,dz',$$

where $G = \{(z,z') \in [a,b]^2\times[a,b]^2\colon z \text{ and } z' \text{ are unordered}\}$. Given processes $\varphi(t,x)$ and $\psi(z,z',x)$ for which $E\int_{[a,b]^d} |\varphi(t)|^2 dt$ and $E\int_{[a,b]^2}\int_{[a,b]^2} |\psi(z,z')|^2 dz\,dz'$ are finite, we prove existence theorems for the white noise integrals $\int_{[a,b]^d} \overset{*}{\partial}_t \varphi(t,x)\,dt$ and $\int_{[a,b]^2}\int_{[a,b]^2} \overset{*}{\partial}_{z'} \overset{*}{\partial}_z 1_G(z,z')\varphi(z,z')\,dz\,dz'$. Here, as in the one–dimensional case, no special measurability restrictions are necessary. We will then define $F(W(s,t))$ for $F \in \mathscr{S}^*(\mathbb{R}^2)$ and $(s,t) \in \mathbb{R}^2_+$ with $st > 0$, and obtain the following Ito formula: For $0 < a_1 < b_1$ and $0 < a_2 < b_2$,

$F(W(b_1,b_2)) - F(W(b_1,a_2) - (F(W(a_1,b_2) - F(W(a_1,a_2)) =$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} \overset{*}{\partial}_{(s,t)} F'(W(s,t))\,ds\,dt$$

$$+\frac{1}{2}\Big[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1} - \int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1} + \int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1} - \int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1}$$
$$\overset{*}{\partial}_{(u,v)} \overset{*}{\partial}_{(s,t)} 1_G((s,t),(u,v))F''(W(s,t)\vee(u,v))\,ds\,dt\,du\,dv\Big]$$

$$+ \frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1,t))\,dt - \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1,t))\,dt$$

$$+ \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2}\int_0^u \overset{*}{\partial}_{(s,t)}{}^{\frac{t}{2}}\, F'''(W(u,t))dsdt\right]du.$$

For $F \in \mathscr{S}(\mathbb{R})$, the formula reduces to the one given by Cairoli and Walsh [2]. If $F(W(s,t))$ is a martingale on increasing paths, the last three integrals on the right side vanish giving the Wong and Zakai formula.

The white noise space $(\mathscr{S}^*,\mu)$ and the Hida theory thus offer a new way to look at stochastic integration, perhaps a simpler view. A natural question for future consideration is whether one can formulate the theory of stochastic differential equations in this setting what the advantage of such a formulation would be.

Chapter 1. Background

§ 1. The Hida theory of generalized Wiener functionals.

In this section we will describe the parts of the Hida theory which are pertinent to our work. The ideas involved in the multidimensional case are natural extensions of those for one dimension. We will, therefore, present these ideas in the more general setting.

For $d \in Z^+$, set $\mathbb{R}_+^d = \{ \mathbf{x} = (x_1,...,x_d) \in \mathbb{R}^d : x_i \geq 0 \}$ and denote by $\mathbb{R}_{\mathbf{x}}$ the rectangle $\prod_{i=1}^{d} [0,x_i[$.

<u>Definition 1.1.1</u>

A Wiener process with d – dimensional time is a system $\{ W(\mathbf{x},\omega) : \mathbf{x} \in \mathbb{R}_+^d \}$ of random variables which satisfy

(a) $W(\mathbf{0},\omega) = 0$ a.e.

(b) For almost ω, $W(\cdot,\omega)$ is continuous

(c) The system $\{ W(\mathbf{x},\omega) \}$ is Gaussian, $W(\mathbf{x},\omega)$ has zero mean

$$\text{and } E(W(\mathbf{x},\omega)W(\mathbf{y},\omega)) = \prod_{j=1}^{d} (x_j \wedge y_j)$$

<u>Note</u> Taking $d = 1$, one gets a Brownian motion

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth real valued functions on $\mathbb{R}^d$. The dual space $\mathscr{S}^*(\mathbb{R}^d)$ of $\mathscr{S}(\mathbb{R}^d)$ consists of the tempered distributions. Thus we have the continuous inclusions $\mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathscr{S}^*(\mathbb{R}^d)$. The cannonical bilinear form connecting $\mathscr{S}$ and $\mathscr{S}^*$ will be denoted by $\langle x,\xi \rangle$, $x \in \mathscr{S}^*$ and $\xi \in \mathscr{S}$. Also, $\mathscr{S}$ is a countably Hilbert nuclear space [9], i.e., $\mathscr{S}$ is topologized by a family $\{\|\cdot\|_p;\ p = 1,2,...\}$ of Hilbertian norms with the following structure: Let $\mathscr{S}_p$ be the completion of $\mathscr{S}$ with respect to the norm $\|\cdot\|_p$. Then

$$\mathscr{S} = \bigcap_p \mathscr{S}_p \subset \cdots \mathscr{S}_2 \subset \mathscr{S}_1 \subset \mathscr{S}_0 = L^2(\mathbb{R}^d) \subset \mathscr{S}_1^* \subset \mathscr{S}_2^* \subset \cdots \subset \bigcup_p \mathscr{S}_p^* = \mathscr{S}^*$$

where the inclusions $\mathscr{S}_{p+1} \subset \mathscr{S}_p$ are Hilbert–Schmidt, and the inclusions $\mathscr{S} \subset \mathscr{S}_p$ and $\mathscr{S}_p^* \subset \mathscr{S}^*$ are continuous. If $H_n(x) = (-1)^n \exp(x^2) D_x^n \exp(-x^2)$, let $h_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) \exp(-x^2/2)$. Then $\{h_n\}_{n=0}^{\infty}$ is a complete orthonormal system in $L^2(\mathbb{R})$ and $\{h_{n_1 n_2 \cdots n_d}\}_{n_1, n_2, \ldots, n_d = 0}^{\infty}$, where $h_{n_1 n_2 \cdots n_d}(t_1, \ldots, t_d) = \prod_{i=1}^{d} h_{n_i}(t_i)$, is a c.o.n.s. in $L^2(\mathbb{R}^d)$. Let us denote this basis by $\{\xi_n\}_{n=0}^{\infty}$. For $f \in \mathscr{S}(\mathbb{R}^d)$ and p an integer, $\|f\|_p^2 = \sum_{n=0}^{\infty} (2n+1)^{2p} (f, \xi_n)^2$, the inner product on $L^2(\mathbb{R}^d)$ being denoted by $(\cdot,\cdot)$. Note that $\mathscr{S}_p = \{f \in L^2(\mathbb{R}^d) : \|f\|_p < \infty\}$. Also, it is true that $\mathscr{S}_p^* = \mathscr{S}_{-p}$.

<u>**Theorem 1.1.1**</u> (Bochner – Minlos)

Let $C(\xi)$ be a functional on $\mathscr{S}(\mathbb{R}^d)$ which is (i) continuous; (ii) positive definite, and; (iii) $C(0) = 1$. Then there exists a unique probability measure μ on $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B})$ such that

$$C(\xi) = \int_{\mathscr{S}^*(\mathbb{R}^d)} \exp(i\langle x, \xi\rangle) \, d\mu(x)$$

Here $\mathscr{B}$ is the σ – field generated by the cylinder subsets of $\mathscr{S}^*(\mathbb{R}^d)$ of the form $\{ x \,;\, (\langle x,\xi_1\rangle, \ldots, \langle x,\xi_n\rangle) \in B$, B a Borel subset of $\mathbb{R}^n$, $\xi_1, \ldots, \xi_n \in \mathscr{S}(\mathbb{R}^d)$, $n = 1,2,\ldots\}$.

<u>**Definition 1.1.2**</u>

The probability space $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B}, \mu)$ determined by the characteristic functional $C(\xi) = \exp(-\frac{1}{2}\|\xi\|^2)$ is called the d – dimensional white noise space. Here, $\|\cdot\|$ is the $L^2(\mathbb{R}^d)$ norm.

For $\xi \in \mathscr{S}$, the random variable $\langle x,\xi\rangle$ on $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B},\mu)$ is normally distributed with mean 0 and variance $\|\xi\|^2$. Moreover, since $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, if $f \in L^2(\mathbb{R}^d)$, $\langle \cdot,f\rangle$ is Gaussian with mean 0 and variance $\|f\|^2$. We thus have that $\{ \langle x,1_A\rangle : A$ is a Borel subset of $\mathbb{R}^d$ with finite Lebesgue measure $\}$ is a normal random measure on $(\mathscr{S}^*(\mathbb{R}^d), \mathscr{B},\mu)$.

<u>Example 1.1.1</u>

For $\mathbf{a} \in \mathbb{R}^d_+$ and $x \in \mathscr{S}^*(\mathbb{R}^d)$, $W(\mathbf{a},x) = \langle x,1_{\mathbb{R}_\mathbf{a}}\rangle$ is a d – dimensional time Wiener process. For $d = 1$ we have the Brownian motion $B(t,x) = \langle x,1_{[0,t[}\rangle$.

<u>Theorem 1.1.2</u> (Wiener – Ito decomposition)

$L^2(\mathscr{S}^*(\mathbb{R}^d))$ has the direct orthogonal decomposition $L^2(\mathscr{S}^*(\mathbb{R}^d)) = \sum_{n=0}^{\infty} \oplus K_n$ where $K_0 = \mathbb{R}$ and for $n > 1$, K_n is the space of n–tuple Wiener integrals based on the normal random measure $W_A = \langle x,1_A\rangle$ mentioned above, i.e., each φ in K_n has the following form

$$\varphi(x) = \int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1, \quad ,\mathbf{u}_n)dW(\mathbf{u}_1,x)...dW(\mathbf{u}_n,x)$$

where $f \in \hat{L}((\mathbb{R}^d)^n)$: the $L((\mathbb{R}^d)^n)$ functions which are symmetric in the $\mathbf{u}_i$'s. Moreover, $(\varphi,\psi)_{L^2(\mathscr{S}^*(\mathbb{R}^d))} = \sqrt{n!}\,(f,g)_{L^2((\mathbb{R}^d)^n)}$, where $\psi \in K_n$ is the multiple Wiener integral of g.

For the proof of this theorem see Ito [8]. It is also shown there that if $\{\eta_i\}_{i=1}^{k}$ is an orthonormal set in $L^2(\mathbb{R}^d)$ and $p_1 + \cdots + p_k = n$, then

$$\int_{(\mathbb{R}^d)^n} \eta_1(\mathbf{u}_1)\cdots\eta_1(\mathbf{u}_{p_1})\eta_2(\mathbf{u}_{p_1+1})\cdots\eta_2(\mathbf{u}_{p_1+p_2})\cdots\eta_k(\mathbf{u}_{p_1+\ldots+p_{k-1}})\cdots \times \eta_k(\mathbf{u}_n)dW(\mathbf{u}_1)\cdots dW(\mathbf{u}_n)$$

$$= (\sqrt{2})^{-n} \prod_{i=1}^{k} H_{p_i}(\int \eta_i dW/\sqrt{2}),$$

where H_{p_i} is the Hermite polynomial of degree p_i. This product is called a Fourier–Hermite polynomial of degree n based on $\{\eta_i\}$. Then for a c.o.n.s. $\{\eta_i\}$ of $L^2(\mathbb{R}^d)$, K_n is spanned by the Fourier–Hermite polynomials of degree n based on $\{\eta_i\}$. Note that for a Borel set A with finite Lebesgue measure $\int_A dW(u) = \langle x, 1_A \rangle$. Thus, for $f \in L^2(\mathbb{R}^d)$, $\langle x,f \rangle = \int_{\mathbb{R}^d} f(u)dW(u)$, and in particular $\int \eta_i dW = \langle x, \eta_i \rangle$.

In the Hida theory, functionals in $L^2(\mathscr{S}^*(\mathbb{R}^d))$ are studied by means of a transformation to a space of functionals on $\mathscr{S}(\mathbb{R}^d)$. Once this transformation is made, the resulting functional has a very nice form which allows us to work in $\sum_{n=0}^{\infty} \oplus \sqrt{n!}\, \hat{L}^2((\mathbb{R}^d)^n)$.

Definition 1.1.3

The S – transform on $L^2(\mathscr{S}^*(\mathbb{R}^d))$ is defined by

$$(S\varphi)(\xi) = \int_{\mathscr{S}^*} \varphi(x + \xi) d\mu(x)$$

and $U(\xi) = (S\varphi)(\xi)$ is called the U – functional of φ. Note that for $\varphi = \sum_{n=0}^{\infty} \varphi_n$, $\varphi_n \in K_n$, $(S\varphi)(\xi) = \sum_{n=0}^{\infty} (S\varphi_n)(\xi)$.

Remark The image of the S – transform is topologized so as to make S a Hilbert space isomorphism , see [5].

<u>Theorem 1.1.3</u> (Integral Representation Theorem)

Suppose $\varphi \in K_n$ is of the form

$$\varphi(x) = \int_{(\mathbb{R}^d)^n} f(u_1,\dots,u_n)dW(u_1,x)\dots dW(u_n,x),$$

with $f \in \hat{L}((\mathbb{R}^d)^n)$. Then

$$(S\varphi)(\xi) = \int_{(\mathbb{R}^d)^n} f(u_1,\dots,u_n)\xi(u_1)\dots\xi(u_n)du_1\dots du_n.$$

The correspondence $\varphi \to f$ is bijective and $\|\varphi\|_{L^2(\mathscr{S}^*(\mathbb{R}^d)} = \sqrt{n!}\|f\|_{L^2((\mathbb{R}^d)^n)}$.

<u>Remark</u> This theorem was stated in [6] without proof. Our proof is based on that of the one–dimensional case found in [5].

<u>Proof</u> First suppose $\varphi(x) = \exp(\sqrt{2}t\langle x,\eta\rangle - t^2)$ with $\eta \in L^2(\mathbb{R}^d)$ such that $\|\eta\| = 1$. Then,

$$\begin{aligned} S\varphi(\xi) &= \int_{\mathscr{S}^*}\exp(\sqrt{2}t\langle x+\xi,\eta\rangle - t^2)d\mu(x) \\ &= \exp(\sqrt{2}t\langle \xi,\eta\rangle - t^2)\int_{\mathscr{S}^*}\exp(\sqrt{2}t\langle x,\eta\rangle)d\mu(x) \\ &= \exp(\sqrt{2}t\langle \xi,\eta\rangle - t^2)\exp(t^2) = \exp(\sqrt{2}t\langle \xi,\eta\rangle) \\ &= \sum_{k=0}^{\infty}((\sqrt{2}t)^k/k!)(\eta,\xi)^k, \ (\cdot,\cdot) \text{ being the inner product in } L^2(\mathbb{R}^d). \end{aligned}$$

However, $\exp(\sqrt{2}t\langle x,\eta\rangle - t^2) = \sum_{k=0}^{\infty}(t^k/k!)H_k(\langle x,\eta\rangle/\sqrt{2})$[5,p311], so that

$$S\varphi(\xi) = \sum_{k=0}^{\infty}(t^k/k!)S\varphi(H_k(\langle x,\eta\rangle/\sqrt{2}))(\xi).$$

Comparing this with the expression above we get that

$$S\varphi(H_k(\langle x,\eta\rangle/\sqrt{2}))(\xi) = (\sqrt{2})^k(\eta,\xi)^k.$$

Let $\{\eta_k\}_{k=1}^{\infty} \subset \mathscr{S}(\mathbb{R}^d)$ be a complete orthonormal system for $L^2(\mathbb{R}^d)$. Then the collection of normalized Fourier–Hermite polynomials $(\prod_j k_j 2^{k_j})^{\frac{1}{2}} \prod_j H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})$ forms a complete orthonormal system in $L^2(\mathscr{S}^*(\mathbb{R}^d))$. Now suppose $\varphi \in K_n$ and $\varphi(x) = \prod_j H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})$, where $n = \sum k_j$. By the remarks following theorem 1.1.2,

$\varphi(x) = \int_{(\mathbb{R}^d)^n}(\sqrt{2})^n \eta_1(u_1)\cdots\eta_1(u_{k_1})\eta_2(u_{k_1+1})\cdots\eta_2(u_{k_1+k_2})\cdots dW(u_1)\cdots dW(u_n)$, and for $\xi \in \mathscr{S}(\mathbb{R}^d)$, $\xi = \sum_j a_j\eta_j + \xi'$ where ξ' is orthogonal to all η_j in the first term. By definition

$$S\varphi(\xi) = \int_{\mathscr{S}^*}\prod_j H_{k_j}(\langle x+\xi,\eta_j\rangle/\sqrt{2})d\mu(x)$$

However, for $p > 1$, the inclusion $L^2(\mathbb{R}^d) \subset \mathscr{S}_p{}^*$ is Hilbert–Schmidt and therefore $(L^2(\mathbb{R}^d),\mathscr{S}^*)$ is an abstract Wiener space (see [14, p59]). We also know from the Bochner–Minlos theorem that μ is concentrated on $\mathscr{S}_p{}^*$. Thus, by the translation formula for abstract Wiener measures [14]: for $f \in L^2(\mathbb{R}^d)$, $d\mu(x+f) = \exp(-\frac{1}{2}\|f\|^2 - \langle x,f\rangle)d\mu(x)$,

$$\begin{aligned} S\varphi(\xi) &= \exp(-\tfrac{1}{2}\|\xi\|^2)\int_{\mathscr{S}^*}\exp(\langle x,\xi\rangle)\prod_j H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})d\mu(x) \\ &= \exp(-\tfrac{1}{2}\|\xi\|^2)\int_{\mathscr{S}^*}\exp(\langle x,\Sigma\, a_j\eta_j\rangle)\exp(\langle x,\xi'\rangle)\prod_j H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})d\mu(x). \end{aligned}$$

Since $(f,g)_{L^2(\mathbb{R}^d)} = 0$ implies that $\langle x,f\rangle$ and $\langle x,g\rangle$ are independent in $L^2(\mathscr{S}^*(\mathbb{R}^d))$, this

$$\begin{aligned} &= \exp(-\tfrac{1}{2}\|\xi\|^2)\int_{\mathscr{S}^*}\exp(\langle x,\xi'\rangle)d\mu(x)\int_{\mathscr{S}^*}\prod_j\exp(\langle x,\Sigma\, a_j\eta_j\rangle)\, H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})d\mu(x) \\ &= \exp(-\tfrac{1}{2}\|\xi\|^2)\exp(\tfrac{1}{2}\|\xi'\|^2)\prod_j\int_{\mathscr{S}^*}\exp(\langle x,\Sigma a_j\eta_j\rangle)\, H_{k_j}(\langle x,\eta_j\rangle/\sqrt{2})d\mu(x) \\ &= \exp(-\tfrac{1}{2}\|\xi\|^2)\exp(\tfrac{1}{2}\|\xi'\|^2)\prod_j\exp(\tfrac{1}{2}a_j^2)\int_{\mathscr{S}^*} H_{k_j}(\langle x+a_j\eta_j,\eta_j\rangle/\sqrt{2})d\mu(x) \\ &= \exp(-\tfrac{1}{2}\|\xi\|^2)\exp(+\tfrac{1}{2}\|\xi'\|^2+\tfrac{1}{2}\Sigma\, a_j^2)\prod_j(\sqrt{2})^{k_j}(\eta_j,a_j\eta_j)^{k_j}. \end{aligned}$$

Noticing that $\|\xi'\|^2+\Sigma\, a_j^2 = \|\xi\|^2$ and that $a_j = (\xi,\eta_j)$, this last expression is

$$\begin{aligned} &= (\sqrt{2})^n\prod_j(\eta_j,\xi)^{k_j} \\ &= \int_{(\mathbb{R}^d)^n}[(\sqrt{2})^n\eta_1(u_1)\cdots\eta_1(u_{k_1})\eta_2(u_{k_1+1})\cdots\eta_2(u_{k_1+k_2})\cdots] \\ &\qquad\qquad \times \xi(u_1)\cdots\xi(u_n)du_1\cdots du_n. \end{aligned}$$

Letting $F(u_1,\dots,u_n)$ be the function in brackets and observing that the value of the integral is unchanged if F is replaced with its symmetrization $\hat{F}$, we have that

$$S\varphi(\xi) = \int_{(\mathbb{R}^d)^n} \hat{F}(u_1,\dots,u_n)\xi(u_1)\cdots\xi(u_n)du_1\cdots du_n.$$

We also have that $\|\hat{F}\|_{L^2((\mathbb{R}^d)^n)} = (\prod_j k_j!)^{\frac{1}{2}}(n!)^{-\frac{1}{2}} 2^{\frac{n}{2}} = (n!)^{-\frac{1}{2}}\|\varphi\|_{L^2(\mathscr{S}^*)}$. Hence, if we have a c.o.n.s. $\{\varphi_m\}$ of Fourier–Hermite polynomials for K_n, then to each φ_m there corresponds a symmetric $L^2((\mathbb{R}^d)^n)$ function $\hat{F}_m$ such that (†) $\|\varphi\|_{L^2(\mathscr{S}^*)} = \sqrt{n!}\ \|\hat{F}\|_{L^2((\mathbb{R}^d)^n)}$ and this collection $\{\hat{F}_m\}$ forms an orthogonal system in $L^2((\mathbb{R}^d)^n)$. With any linear combination of the $\{\varphi_m\}$ we can associate the corresponding linear combination of the $\{\hat{F}_m\}$ so that (†) can be extended to any $\varphi \in K_n$.

Generalized Wiener functionals arise in the following way. For $\alpha \in \mathbb{R}$, let $H^{\alpha}(\mathbb{R}^{nd})$ be the Sobolev space of order α over $\mathbb{R}^{nd}$, i e., $H^{\alpha}(\mathbb{R}^{nd}) = \{f \in \mathscr{S}^*(\mathbb{R}^{nd}) : \int_{\mathbb{R}^{nd}} (1 + |\lambda|^2)^{\alpha}|(\mathscr{F}f)(\lambda)|^2 d\lambda < \infty\}$, $\mathscr{F}$ being the Fourier transform. This is a Hilbert space with dual $H^{-\alpha}(\mathbb{R}^{nd})$.

Define $K_n^{(n)}$ to be the elements of K_n which are n – tuple Wiener integrals of functions in $\hat{H}^{\left(\frac{nd+1}{2}\right)}(\mathbb{R}^{nd})$. We then have the following diagram:

$$\begin{array}{ccccc}
K_n^{(n)} & \hookrightarrow & K_n & \hookrightarrow & K_n^{(-n)} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\sqrt{n!}\ \hat{H}^{\frac{nd+1}{2}}(\mathbb{R}^{nd}) & \hookrightarrow & \sqrt{n!}\ \hat{L}^2((\mathbb{R}^d)^n) & \hookrightarrow & \sqrt{n!}\ \hat{H}^{-\left(\frac{nd+1}{2}\right)}(\mathbb{R}^{nd})
\end{array}$$

$K_n^{(-n)}$ is defined as the space of generalized n–tuple Wiener integrals of elements in $\hat{H}^{-\left(\frac{nd+1}{2}\right)}(\mathbb{R}^{nd})$. Here $\hat{H}^{\left(\frac{nd+1}{2}\right)}(\mathbb{R}^{nd}) = H^{\frac{nd+1}{2}}(\mathbb{R}^{nd}) \cap \hat{L}^2((\mathbb{R}^d)^n)$. More precisely, for

$\varphi = \int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n)dW(u_1)\ldots dW(u_n)$, $f \in \hat{L}^2((\mathbb{R}^d)^n)$, define $|||\varphi||| = (\sqrt{n!})\|f\|_{H^{-\frac{(nd+1)}{2}}(\mathbb{R}^{nd})}$ and let $K_n^{(-n)}$ be the completion of K_n with respect to $|||\cdot|||$ We will write formally that for $\varphi \in K_n^{(-n)}$,

$$\varphi(x) = \int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n)dW(u_1,x)\ldots dW(u_n,x), \text{ where } f \in \hat{H}^{-\frac{(nd+1)}{2}}(\mathbb{R}^{nd}).$$

$K_n^{(-n)}$ can be viewed as the dual of $K_n^{(n)}$ with the pairing $\langle\cdot,\cdot\rangle$: For $\varphi \in K_n^{(n)}$ represented by $f \in \hat{H}^{\frac{(nd+1)}{2}}(\mathbb{R}^{nd})$ and $\psi \in K_n^{(-n)}$ represented by $g \in \hat{H}^{-\frac{(nd+1)}{2}}(\mathbb{R}^{nd})$, $\langle\psi,\varphi\rangle = n!\langle g,f\rangle$.

<u>Remark</u> The S – transform also makes sense for $\varphi \in (L^2)^-$.

<u>Example 1.1.2</u>

Set $\Delta = (\epsilon,\ldots,\epsilon)$, $\epsilon > 0$, and $\mathbf{t} = (t_1,\ldots t_d) \in \mathbb{R}_+^d$. Let

$$\varphi_\Delta = \frac{W(\mathbf{t}+\Delta,x) - W(\mathbf{t},x)}{\epsilon^d} = \frac{1}{\epsilon^d}\langle x, 1_{[\mathbf{t},\mathbf{t}+\Delta[}\rangle$$

where$[\mathbf{t},\mathbf{t}+\Delta[= \{\mathbf{x} \in \mathbb{R}_+^d : t_i \le x_i < t_i + \epsilon;\ 1 \le i \le d\}$. Then

$$(S\varphi_\Delta)(\xi) = \frac{1}{\epsilon^d}\int_{t_d}^{t_d+\epsilon}\cdots\int_{t_1}^{t_1+\epsilon}\xi(s_1,\ldots,s_d)ds_1\ldots ds_d \longrightarrow \xi(t_1,\ldots,t_d)$$

as $\Delta \to 0$, and $\frac{1}{\epsilon^d} 1_{[\mathbf{t},\mathbf{t}+\Delta[} \longrightarrow \delta_{\mathbf{t}}$ in $H^{-\frac{(d+1)}{2}}(\mathbb{R}^d)$. Thus $\lim_{\Delta\to 0} \varphi_\Delta \in K_1^{(-1)}$ and may be written as $\int_{\mathbb{R}^d} \delta_{\mathbf{t}}(\mathbf{u})dW(\mathbf{u})$. We will also denote this limit as $\dot{W}(\mathbf{t})$.

<u>Example 1.1.3</u>

Let $\varphi_\Delta = \left[\frac{W(\mathbf{t}+\Delta,x) - W(\mathbf{t},x)}{\epsilon^d}\right]^2 = \frac{1}{\epsilon^{2d}}\langle x, 1_{[\mathbf{t},\mathbf{t}+\Delta[}\rangle^2$ which is in K_2.

Then

$$(S\varphi)(\xi) = \frac{1}{\epsilon^{2d}} \int_{\mathscr{S}^*} \langle x + \xi, 1_{[t,t+\Delta[}\rangle^2 d\mu(x)$$

$$= \frac{1}{\epsilon^{2d}}(\epsilon^d + \langle \xi, 1_{[t,t+\Delta[}\rangle^2) = \frac{1}{\epsilon^d} + \left[\frac{1}{\epsilon^d}\langle \xi, 1_{[t,t+\Delta[}\rangle\right]^2.$$

Consider $\psi_\Delta = \varphi_\Delta - \frac{1}{\epsilon^d}$, then

$$(S\psi)(\xi) = \int_{(\mathbb{R}^d)^2} \frac{1}{\epsilon^d} 1_{[t,t+\Delta[}(u_1)\frac{1}{\epsilon^d} 1_{[t,t+\Delta[}(u_2)\xi(u_1)\xi(u_2)du_1du_2 \longrightarrow \xi(t)^2$$

as $\Delta \to 0$, and $\frac{1}{\epsilon^d} 1_{[t,t+\Delta[} \frac{1}{\epsilon^d} 1_{[t,t+\Delta[} \longrightarrow \delta_t \hat{\otimes} \delta_t$ in $\hat{H}^{-\frac{(2d+1)}{2}}(\mathbb{R}^{2d})$. The $\lim_{\Delta\to 0} \psi_\Delta$

is then in $K_2^{(-2)}$ and is denoted as $:\dot{W}(t)^2:$ or $\int_{(\mathbb{R}^d)^2} \delta_t \hat{\otimes} \delta_t(u,v)dW(u)dW(v)$.

In general, for $H_n(x,\sigma^2) = \frac{1}{n!}(-\sigma^2)^n \exp(-x^2/2\sigma^2)D_x^n \exp(-x^2/2\sigma^2)$, consider

$\varphi_\Delta = n!H_n\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d}, \frac{1}{\epsilon^d}\right]$, $n > 2$, which is in K_n. In fact φ_Δ is the

multiple Wiener integral of $\frac{1}{(\epsilon^d)^n} 1_{[t,t+\Delta[}(u_1) \cdots 1_{[t,t+\Delta[}(u_n)$. To see this use the fact

that $H_n(x,\sigma) = \frac{x}{n} H_{n-1}(x,\sigma) - \frac{\sigma}{n} H_{n-1}(x,\sigma)$ [5, p313]. Then

$$\varphi_\Delta = n!\frac{1}{n} \frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d} H_{n-1}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d}, \frac{1}{\epsilon^d}\right]$$

$$- n! \frac{1}{n} \frac{1}{\epsilon^d} H_{n-2}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d}, \frac{1}{\epsilon^d}\right].$$

By induction this equals

$$n!\left[\frac{1}{n}\int_{\mathbb{R}^d}\frac{1}{\epsilon^d}1_{[t,t+\Delta[}(u_1)\,dW(u_1)\frac{1}{(n-1)!}\int_{(\mathbb{R}^d)^{n-1}}\frac{1}{(\epsilon^d)^{n-1}}1_{[t,t+\Delta[}(u_1)\cdots 1_{[t,t+\Delta[}(u_{n-1})\right.$$

$$\left.\times\, dW(u_1)\cdots dW(u_{n-1})\right] - n!\,\frac{1}{n}\frac{1}{\epsilon^d}\,H_{n-2}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d},\frac{1}{\epsilon^d}\right].$$

Recall the product formula for multiple Wiener integrals [8]:

$$\int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n)dW(u_1)\cdots dW(u_n)\int_{(\mathbb{R}^d)^n} g(u_1)dW(u_1) =$$
$$\int_{(\mathbb{R}^d)^n}(f\hat{\otimes}g)(u_1,\ldots,u_{n+1})dW(u_1)\cdots dW(u_{n+1})$$
$$+\, n\int_{(\mathbb{R}^d)^{n-1}}\left(\int_{\mathbb{R}^d} f(u_1,\ldots,u_n)g(u_n)du_n\right)dW(u_1)\cdots dW(u_{n-1}).$$

Here $f \in \hat{L}^2((\mathbb{R}^d)^n)$ and $f\hat{\otimes}g$ is the symmetrization of $f\otimes g$. Thus

$$\varphi_\Delta = \int_{(\mathbb{R}^d)^n}\frac{1}{(\epsilon^d)^n}1_{[t,t+\Delta[}(u_1)\cdots 1_{[t,t+\Delta[}(u_n)\,dW(u_1)\cdots dW(u_n) +$$

$$(n-1)\frac{1}{\epsilon^d}\int_{(\mathbb{R}^d)^{n-2}}\frac{1}{(\epsilon^d)^{n-2}}1_{[t,t+\Delta[}(u_1)\cdots 1_{[t,t+\Delta[}(u_{n-2})\,dW(u_1)\cdots dW(u_{n-2})$$

$$-\,(n-1)!\frac{1}{\epsilon^d}\,H_{n-2}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d},\frac{1}{\epsilon^d}\right]$$

$$= \int_{(\mathbb{R}^d)^n}\frac{1}{(\epsilon^d)^n}1_{[t,t+\Delta[}(u_1)\cdots 1_{[t,t+\Delta[}(u_n)\,dW(u_1)\cdots dW(u_n)$$

$$+\,(n-1)\frac{1}{\epsilon^d}(n-2)!\,H_{n-2}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d},\frac{1}{\epsilon^d}\right]$$

$$-\,(n-1)\frac{1}{\epsilon^d}(n-2)!\,H_{n-2}\left[\frac{W(t+\Delta,x) - W(t,x)}{\epsilon^d},\frac{1}{\epsilon^d}\right]$$

$$= \int_{(\mathbb{R}^d)^n} \frac{1}{(\epsilon^d\ )^n}\, 1_{[t,t+\Delta[}(u_1) \cdots 1_{[t,t+\Delta[}(u_n)\ dW(u_1)\cdots dW(u_n).$$

Therefore, $(S\varphi)(\xi) = \left[\frac{1}{\epsilon^d}\int_{[t,t+\Delta[}\xi(u)du\right]^n$

$$= \int \cdots \int_{(\mathbb{R}^d)^n} \frac{1}{\epsilon^d} 1_{[t,t+\Delta[}(u_1) \cdots \frac{1}{\epsilon^d} 1_{[t,t+\Delta[}(u_n)\xi(u_1)\ldots\xi(u_n)du_1\ldots du_n$$

and $\hat{\otimes}_n \frac{1}{\epsilon^d} 1_{[t,t+\Delta[} \to \hat{\otimes}_n \delta_t$ in $\hat{H}^{-\left(\frac{nd+1}{2}\right)}(\mathbb{R}^{nd})$. Thus $\lim_{\Delta\to 0} \varphi_\Delta$ is in $K_n^{(-n)}$ and is represented by $\hat{\otimes}_n \delta_t$. This limit is denoted $:\dot{W}(t)^n:$.

Define $(L^2)^+ = \sum_{n=0}^{\infty} \oplus K_n^{(n)}$ and $(L^2)^- = \sum_{n=0}^{\infty} \oplus K_n^{(-n)}$ where $K_0^{(0)}$ is the real number system. Thus we have

$$(L^2)^+ \subset L^2(\mathscr{S}^*) \subset (L^2)^-.$$

$(L^2)^+$ is called the space of test functionals and $(L^2)^-$ is called the space of generalized functionals. For $\varphi = \sum_{n=0}^{\infty} \varphi_n$ in $(L^2)^+$ and $\psi = \sum_{n=0}^{\infty} \psi_n$ in $(L^2)^-$,

$\langle\varphi,\psi\rangle = \sum_{n=0}^{\infty} \langle\varphi_n,\psi_n\rangle.$

In the Hida theory $\{\dot{B}(t),\ t\in\mathbb{R}\}$ is viewed as a coordinate system in $\mathscr{S}^*$ so as to take time into account, recall (example 1.1.2) that $\dot{B}(t) = \delta_t$ is in $K_1^{(-1)}$. Derivatives are then taken with respect to this coordinate system. This idea can be carried over to higher dimensions.

Definition 1.1.3

Suppose U is a functional on $\mathscr{S}(\mathbb{R}^d)$. The first variaton of U at ξ is a functional δU_ξ on $\mathscr{S}$ such that

(i) δU_ξ is continuous and linear on $\mathscr{S}$

(ii) For every $p > 0$, $U(\xi + \eta) - U(\xi) - \delta U_\xi(\eta) = o(\|\eta\|_p)$ as $\eta \to 0$.

Remark If δU_ξ exists then $\delta U_\xi(\eta) = \lim_{\lambda \to 0} \frac{U(\xi + \lambda\eta) - U(\xi)}{\lambda}$.

Definition 1.1.4

Let U be the U–functional of $\varphi \in (L^2)^-$. Suppose the first variation of U is given by

$$(\delta U)_\xi(\eta) = \int_{\mathbb{R}^d} U'(\xi,u)\eta(u)du\ ;\ \eta \in \mathscr{S}(\mathbb{R}^d).$$

If $U'(\xi,\cdot)$ is an ordinary function and $U'(\cdot,t)$ is a U–functional, then $\partial_t\varphi$ is defined to be the generalized Wiener functional with U–functional $U'(\cdot,t)$, i.e.,

$$U(\partial_t\varphi)(\xi) = U'(\xi,t).$$

Example 1.1.4 Let $\varphi(x) = \langle x,\xi_0\rangle \in K_1$ where $\xi_0 \in \mathscr{S}(\mathbb{R}^d)$. Then $U\varphi(\xi) = \langle \xi,\xi_0\rangle$ and $\delta U_\xi(\eta) = \langle \eta,\xi_0\rangle$. Thus $U'(\xi,u) = \xi_0(u)$ and $\partial_t\varphi(x) = \xi_0(t) \in K_0$.

Example 1.1.5 Let $\varphi(x) = \langle x,\xi_0\rangle\langle x,\xi_1\rangle$, ξ_0, $\xi_1 \in \mathscr{S}(\mathbb{R}^d)$. Then $\varphi \in K_2$ and $U\varphi(\xi) = \langle \xi,\xi_0\rangle\langle \xi,\xi_1\rangle$ and $\delta U_\xi(\eta) = \langle \xi,\xi_0\rangle\langle \eta,\xi_1\rangle + \langle \eta,\xi_0\rangle\langle \xi,\xi_1\rangle = \langle \eta,\langle \xi,\xi_0\rangle\xi_1(u) + \xi_0(u)\langle \xi,\xi_1\rangle\rangle$. Thus $U'(\xi,t) = \langle \xi,\xi_0\rangle\xi_1(t) + \xi_0(t)\langle \xi,\xi_1\rangle$ and $\partial_t\varphi = \langle x,\xi_0\rangle\xi_1(t) + \xi_0(t)\langle x,\xi_1\rangle \in K_1$.

Example 1.1.6 For $\xi_0 \in \mathscr{S}(\mathbb{R}^d)$ let $\varphi(x) = (\sqrt{2})^{-n}H_n(\langle x,\xi_0\rangle/\sqrt{2}) \in K_n$. $U\varphi(\xi) = \langle \xi,\xi_0\rangle^n$ and $\delta U_\xi(\eta) = n\langle \eta,\xi_0\rangle\langle \xi,\xi_1\rangle^{n-1}$, giving that $\partial_t\varphi(x) = (\sqrt{2})^{n-1}n\xi_0(t)H_{n-1}(\langle x,\xi_0\rangle/\sqrt{2})$ which is an element of K_{n-1}.

Let's consider $\varphi = \dot{W}(s) = \delta_s \in K_1^{(-1)}$. We know that $U\varphi(\xi) = \xi(s)$. Thus $\delta U_\xi(\eta) = \eta(s) = \int_{\mathbb{R}^d} \delta_s(u)\eta(u)du$. Even though δ_s is not an ordinary function, we can view it as a regular function into the nonstandard real number system:

$$\delta_s(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} 1_{[s,s+\Delta[}(t) = \begin{cases} 0 & t \neq s \\ 1/dt := (1/dt)^d & t = s \end{cases}$$

If we take K_0 to be the nonstandard real number system, $\partial_t \dot{W}(s)$ can then be defined as $\delta_s(t) \in K_0$. If $d = 1$, this is the infinite dimensional analogue of $\partial_i x_j = \delta_{ij}$ in the finite dimensional case and Kuo [18] has used this idea to differentiate a certain class of generalized functionals without using the S–transform.

Theorem 1.1.4

$$\partial_t : K_n^{(n)} \longrightarrow K_{n-1}^{(n-1)} \text{ and for } \varphi = \int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n) dW(u_1)\ldots dW(u_n)$$

$$\partial_t \varphi = n \int_{(\mathbb{R}^d)^{n-1}} f(t,u_1,\ldots,u_{n-1}) dW(u_1)\ldots dW(u_{n-1})$$

proof Since $f \in \hat{H}^{\frac{nd+1}{2}}(\mathbb{R}^{nd})$, f is continuous by Sobolev's lemma. Also,

$$U\varphi(\xi) = \int_{(\mathbb{R}^d)^n} f(u_1,u_2,\ldots,u_n)\xi(u_1)\cdots\xi(u_n)du_1\ldots du_n.$$

Thus,

$$\delta U_{\xi}(\eta) = \lim_{\lambda \to 0} \frac{1}{\lambda}\Big[\int_{(\mathbb{R}^d)^n} f(u_1,u_2,\ldots u_n)(\xi(u_1) + \lambda\eta(u_1))\cdots(\xi(u_n) + \lambda\eta(u_n))du_1\ldots du_n$$
$$- \int_{(\mathbb{R}^d)^n} f(u_1,u_2,\ldots,u_n)\xi(u_1)\cdots\xi(u_n)du_1\ldots du_n\Big]$$

$$= \sum_{k=1}^{n} \int_{(\mathbb{R}^d)^n} f(u_1,u_2,\ldots,u_n)\xi(u_1)\cdots\widetilde{\xi(u_k)}\cdots\xi(u_n)\eta(u_k)du_1\ldots du_n$$

$\sim$ indicating that this factor is missing. By the symmetry of f this

$$= \int_{\mathbb{R}^d}\Big[n\int_{(\mathbb{R}^d)^{n-1}} f(u_1,u_2,\ldots,u_n)\xi(u_2)\cdots\xi(u_n)du_2\ldots du_n\Big]\eta(u_1)du_1.$$

Thus $U'(\xi,t) = n\int_{(\mathbb{R}^d)^{n-1}} f(t,u_1,\ldots,u_{n-1})\xi(u_1)\cdots\xi(u_{n-1})du_1\ldots du_{n-1}$ and

$$\partial_t \varphi = n\int_{(\mathbb{R}^d)^{n-1}} f(t,u_1,\ldots,u_{n-1})dW(u_1)\ldots dW(u_{n-1}).$$

Definition 1.1.5

Define the adjoint ∂_t^* of ∂_t by $\langle \partial_t^* \psi, \varphi \rangle = \langle \psi, \partial_t \varphi \rangle$, $\psi \in (L^2)^-$, $\varphi \in (L^2)^+$.

Theorem 1.1.5

$\partial_t^*: K_n^{(-n)} \longrightarrow K_{n+1}^{-(n+1)}$ and for $\varphi = \int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n) dW(u_1)\ldots dW(u_n)$,

$$\partial_t^* \varphi = \int_{(\mathbb{R}^d)^{n+1}} (\delta_t \hat{\otimes} f)(u_1,\ldots,u_{n+1}) dW(u_1)\ldots dW(u_{n+1}).$$

proof Let $\psi \in K_{n+1}^{(n+1)}$ be represented by $g(u_1,\ldots,u_{n+1})$. Then

$$\begin{aligned}\langle \partial_t^* \varphi, \psi \rangle = \langle \varphi, \partial_t \psi \rangle &= n! \int_{(\mathbb{R}^d)^n} f(u_1,\ldots,u_n)(n+1) g(t,u_1,\ldots u_n) du_1 \ldots du_n \\ &= (n+1)! \int_{(\mathbb{R}^d)^{n+1}} \delta_t(u_1) f(u_2,\ldots,u_n) g(u_1,u_2,\ldots u_n) du_1 \ldots du_{n+1} \\ &= \langle \int_{(\mathbb{R}^d)^{n+1}} \delta_t \otimes f(u_1,\ldots,u_{n+1}) dW(u_1)\ldots dW(u_{n+1}), \psi \rangle.\end{aligned}$$

Remark For $\varphi \in K_n^{(-n)}$,

$U(\partial_t^* \varphi)(\xi) = \int_{(\mathbb{R}^d)^{n+1}} \delta_t \otimes f(u_1,\ldots,u_{n+1}) \xi(u_1) \cdots \xi(u_{n+1}) du_1 \ldots du_{n+1} = \xi(t) U\varphi(\xi)$.

For $\varphi = \sum_{n=0}^{\infty} \varphi_n$ in $(L^2)^-$, $U(\partial_t^* \varphi)(\xi) = \sum_{n=0}^{\infty} U(\partial_t^* \varphi_n)(\xi) = \sum_{n=0}^{\infty} \xi(t) U\varphi_n(\xi) = \xi(t) U\varphi(\xi)$.

Definition 1.1.6

If (i) for each $t \in [a,b]^d \subset \mathbb{R}_+^d$, $\varphi(t) \in (L^2)^-$, (ii) for each $\xi \in \mathscr{S}$, $U(\varphi(t))(\xi)$ is an integrable function of t on $[a,b]^d$, and (iii) $\xi \to \int_{[a,b]^d} U(\varphi(t))(\xi) dt$ is a U–functional. Then the white noise integral $\int_{[a,b]^d} \varphi(t) dt$ is defined by:

$$U\left[\int_{[a,b]^d} \varphi(t) dt\right](\xi) = \int_{[a,b]^d} U(\varphi(t))(\xi) dt.$$

Note One can similarly define $\int_{([a,b]^d)^n} \varphi(t_1,t_2,\ldots,t_n)dt_1dt_2\cdots dt_n$.

Example 1.1.7 Suppose that

$$\varphi(t) = \int_{(\mathbb{R}^d)^n} f(t,u_1,\ldots,u_n)dW(u_1)\ldots dW(u_n), \text{ where f is in } L^2(([a,b]^d)^{n+1}).$$

Then $\overset{*}{\partial}_t\varphi(t) = \int_{(\mathbb{R}^d)^{n+1}} (\delta_t\hat{\otimes} f)(u_1,\ldots,u_{n+1})dW(u_1)\ldots dW(u_{n+1})$. Now, $U\varphi(t)(\xi) = \int_{(\mathbb{R}^d)^n} f(t,u_1,\ldots,u_n)\xi(u_1)\ldots\xi(u_n)du_1\ldots du_n$, and $U(\overset{*}{\partial}_t\varphi(t))(\xi) = \xi(t)U\varphi(t)(\xi)$ giving that

$$\begin{aligned}\int_{[a,b]^d} U(\overset{*}{\partial}_t\varphi(t))(\xi)dt &= \int_{[a,b]^d} \xi(t)\int_{(\mathbb{R}^d)^n} f(t,u_1,\ldots,u_n)\xi(u_1)\ldots\xi(u_n)du_1\ldots du_n dt \\ &= \int_{(\mathbb{R}^d)^{n+1}} 1_{[a,b]^d}(t)f(t,u_1,\ldots,u_n)\xi(u_1)\ldots\xi(u_n)dtdu_1\ldots du_n.\end{aligned}$$

Letting $h(u_1,\ldots,u_{n+1})$ be the symmetrization of $1_{[a,b]^d}(u_1)f(u_1,\ldots,u_{n+1})$, we then have that

$$\begin{aligned}\int_{[a,b]^d} U(\overset{*}{\partial}_t\varphi(t))(\xi)dt &= \int_{(\mathbb{R}^d)^{n+1}} h(u_1,\ldots,u_{n+1})\xi(u_1)\ldots\xi(u_{n+1})du_1\ldots du_{n+1} \\ &= U\left[\int_{(\mathbb{R}^d)^{n+1}} h(u_1,\ldots,u_{n+1})dW(u_1)\ldots dW(u_{n+1})\right](\xi).\end{aligned}$$

Thus,

$$\int_{[a,b]^d} \overset{*}{\partial}_t\varphi(t)dt = \int_{(\mathbb{R}^d)^{n+1}} h(u_1,\ldots,u_{n+1})dW(u_1)\ldots dW(u_{n+1}).$$

By the following theorem, for $d = 1$, the Brownian functional $\int_{[a,b]^d} \overset{*}{\partial}_t\varphi(t)dt$ is an extension of the Ito stochastic integral.

Theorem 1.1.6 [12]

Suppose $\varphi(t, x)$, $x \in \mathscr{S}^*$, is a non–anticipating stochastic process such that $E\int_a^b |\varphi(t)|^2 dt < \infty$. Then

$$\int_a^b \varphi(t) dB(t) = \int_a^b \partial_t^* \varphi(t) dt$$

where the integral on the left is the Ito integral.

We have the following existence theorems for 1 – dimensional time.

Theorem 1.1.7 [17]

Suppose $\varphi(t)$ is a stochastic process on $(\mathscr{S}^*, \mu)$ such that $E\int_a^b |\varphi(t)|^2 dt < \infty$ and $E\int_a^b\int_a^b |\partial_s\varphi(t)\overline{\partial_t\varphi(s)}| dtds < \infty$. Then $\int_a^b \partial_t^* \varphi(t) dt$ exists and $E|\int_a^b \partial_t^* \varphi(t) dt|^2 = E\int_a^b |\varphi(t)| dt + E\int_a^b\int_a^b |\partial_s\varphi(t)\overline{\partial_t\varphi(s)}| dtds$.

Theorem 1.1.8 [7]

Assume $\varphi(t)$ is a stochastic process on $(\mathscr{S}^*, \mu)$ such that $E\int_a^b |\varphi(t)|^2 < \infty$ and $E\int_a^b\int_a^b |\partial_s\varphi(t)|^2 dtds < \infty$. Then $\int_a^b \partial_t^* \varphi(t) dt$ exists and

$$E|\int_a^b \partial_t^* \varphi(t) dt|^2 \leq E\int_a^b |\varphi(t)|^2 + E\int_a^b\int_a^b |\partial_s\varphi(t)|^2 dtds.$$

Theorem 1.1.9 [7]

Suppose $\varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t)$, $\varphi(t) \in K_n$, and $\sum_{n=0}^{\infty} (n+1) E\int_a^b |\varphi_n(t)|^2 dt < \infty$. Then $\int_a^b \partial_t^* \varphi(t) dt$ exists and $E|\int_a^b \partial_t^* \varphi(t) dt|^2 \leq \sum_{n=0}^{\infty} (n+1) E\int_a^b |\varphi_n(t)|^2 dt$.

We also have

Theorem 1.1.10 [11]

Suppose F is in $\mathscr{S}^*(\mathbb{R})$. Then for any $t > 0$, $F(B(t))$ is in $(L^2)^-$ with U–functional given by

$$U(\xi) = g_t * F(\langle \xi, 1_{[0,t)} \rangle)$$

where $g_t(y) = (2\pi t)^{-\frac{1}{2}} \exp(-y^2/2t)$.

Theorem 1.1.11 [11]

For any $F \in \mathscr{S}^*(\mathbb{R})$ and $0 < s < t$,

$$F(B(t)) - F(B(s)) = \int_s^t \overset{*}{\partial}_u F'(B(u))du + \tfrac{1}{2}\int_s^t F''(B(u))du.$$

§ 2. Stochastic integrals of processes with multidimensional time parameter.

Let $<$ be the partial ordering on $\mathbb{R}_+^d$ where $(x_1,...,x_d) \leq (y_1,...,y_d)$ if and only if $x_i \leq y_i$; $1 \leq i \leq d$. Assume $\{W(\mathbf{x},\omega);\ \mathbf{x} \in \mathbb{R}_+^d\}$ is a Wiener process on a probability space $\{\Omega, \mathscr{F}, P\}$ and denote by $\mathscr{F}_{\mathbf{a}}$ the σ–field generated by $\{W(\mathbf{x});\ \mathbf{x} < \mathbf{a}\}$. Notice that for $\mathbf{a} < \mathbf{b}$, $\mathscr{F}_{\mathbf{a}} \subset \mathscr{F}_{\mathbf{b}}$ and $E(W(\mathbf{b}) \mid \mathscr{F}_{\mathbf{a}}) = W(\mathbf{a})$. $\{\mathscr{F}_{\mathbf{a}};\ \mathbf{a} \in \mathbb{R}_+^d\}$ is said to be an increasing sequence of σ–fields and $\{W(\mathbf{a});\ \mathscr{F}_{\mathbf{a}})$ is called a martingale with respect to this partial ordering.

Let $T = [a,b] \subset \mathbb{R}_+^1$ and consider the Wiener process $\{W(\mathbf{x}); \mathscr{F}_{\mathbf{x}},\ \mathbf{x} \in T^d\}$. Cairoli [1] defined the following Ito–type stochastic integral for d =2 and Wong and Zakai [20] extended the definition to any d. Assume $\varphi(\mathbf{x},\omega)$ satisfies the following conditions:

(1) $\varphi(\mathbf{x},\omega)$ is a bimeasurable function of $(\mathbf{x},\omega)$ with respect to $\mathscr{G} \otimes \mathscr{F}$ where $\mathscr{G}$ denotes the σ–field of Borel sets in T^d

(2) For each $\mathbf{x} \in T^d$, $\varphi(\mathbf{x},\omega)$ is $\mathscr{F}_{\mathbf{x}}$–measurable

(3) $\int_{T^d} E\varphi^2(\mathbf{x},\omega)d\mathbf{x} < \infty$.

<u>Note</u> A process with property (b) is said to be non–anticipating.

Suppose that φ is simple, i.e., $\varphi(x,\omega) = \varphi_v(\omega)$, $x \in \Delta_v$, $v = 1,2,\ldots,k$, and $\varphi = 0$ elsewhere, and that Δ_v are disjoint rectangles $\Delta_v = \prod_{i=1}^{d} [a_i^v, b_i^v[\subset T^d$. Then

$$I_1(\varphi)(\omega) = \int_{T^d} \varphi(x,\omega)dW(x,\omega) := \sum_v \varphi_v(\omega)\Delta_v W(\omega)$$

where for a rectangle $\Delta = \prod_{i=1}^{d} [a_i,b_i[$, $\Delta W(\omega) = \sum_x (-1)^{\pi(x)} W(x,\omega)$, the sum being taken over the 2^d vertices $\{x;\, x_i = a_i \text{ or } b_i\}$ and $\pi(x)$ is the number of b_i's in x. For example, if $d = 2$, $\Delta W = W(b_1,b_2) - W(b_1,a_2) + W(a_1,a_2) - W(a_1,b_2)$. The definition of $I_1(\varphi) = \int_{T^d} \varphi(x)dW(x)$ is then extended to φ satisfying properties (1)–(3) by a standard completion argument. Principal properties of this integral are

<u>Theorem 1.2.1</u>

(a) $\int_{T^d} (\alpha\varphi(x) + \beta\psi(x))dW(x) = \alpha\int_{T^d} \varphi(x)dW(x) + \beta\int_{T^d} \psi(x)dW(x)$

(b) $E(\int_{T^d} \varphi(x)dW(x)\int_{T^d} \psi(x)dW(x)) = E(\int_{T^d} \varphi(x)\psi(x)dx)$

(c) $E(\int_{T^d} \varphi(y)dW(y) \mid \mathcal{F}_x) = \int_{[(a,\ldots,a),x]} \varphi(y)dW(y)$, where $[(a,\ldots,a),x]$ denotes $\{y \in T^d;\, (a,\ldots,a) < y < x\}$.

Wong and Zakai [20] have defined a second type of stochastic integral for $d = 2$, which we will now describe. Both of these integrals are necessary to represent square integrable functionals of $\{W(z); z \in T^2\}$.

For (s,t), (u,v) $\in T^2$ we will use $(s,t)\vee(u,v)$ to denote $(\max\{s,u\},\max\{t,v\})$. Let $G = \{(z,z') \in T^2\times T^2;\ z \text{ and } z' \text{ are unordered}\}$. Suppose that $\psi(\omega,z,z')$ is a function defined on $\Omega\times T^2\times T^2$ satisfying

(1) $\psi(\omega,z,z')$ is jointly measurable with respect to $\mathscr{F}\otimes \mathscr{G} \otimes \mathscr{G}$

(2) For each $z,z' \in T^2$, the function $\psi(\omega,z,z')$ is measurable with respect to $\mathscr{F}_{z\vee z'}$

(3) $E\int_{T^2\times T^2}\psi^2(z,z')dzdz' < \infty$.

Assume that $\psi(\omega,z,z')$ is simple: $\psi(\omega,z,z') = \alpha(\omega)$ for $z \in \Delta_1$ and $z' \in \Delta_2$ and zero elsewhere. For $n = 1,2,\ldots$, partition T into segments of length $(b-a)/2^n$ and let P_n be the partition induced on T^2, with partition points $\{z_{ij}\}_{i,j=1}^{2^n-1}$, and let $\Delta_{ij} = [z_{ij},z_{i+1,j+1}[$. For $\Delta_{ij}W = W(z_{i+1,j+1}) - W(z_{i+1,j}) + W(z_{ij}) - W(z_{i,j+1})$, define

$$I_2^n(\psi) = \sum_{\substack{i,j=1\\k,m=1}}^{2^n-1} \psi(z_{ij},z_{km})1_G(z_{ij},z_{km})\Delta_{ij}W\,\Delta_{km}W$$

It is shown in [20], that $I_2^n(\psi)$ converges in $L^2(\Omega)$ as $n \to \infty$. The integral is then defined to be this limit:

$$I_2(\psi) = \left[\int_{T^2\times T^2}\right]\psi(z,z')dzdz' := \lim_{n\to\infty} \text{ in q.m.} I_2^n(\psi)$$

Note that if $\Delta_1\times \Delta_2 \subset G$, then $I_2(\psi) = \alpha\Delta_1W\,\Delta_2W$. The definition can now be extended to all functions satisfying the above conditions by approximating with linear combinations of simple functions. The main properties of this integral are

<u>Theorem 1.2.2</u>

(a) $I_2(a\varphi + b\psi) = aI_2(\varphi) + bI_2(\psi)$

(b) $I_2(\psi) = I_2(1_G\psi)$

(c) $E(I_2(\psi)I_2(\varphi)) = E\int_{T^2\times T^2} 1_G(z,z')\psi(z,z')\varphi(z,z')dzdz'$

(d) $E(I_1(\varphi)I_2(\psi)) = 0$

(e) $E(I_2(\psi)\,|\,\mathscr{F}_z) = \left[\int_{[(a,a),z[\times[(a,a),z[}\right]\psi(u,u')dW(u)dW(u')$

<u>Theorem 1.2.3</u> (Ito formula) [20]

Define a martingale $M_z = (M_{1z},\ldots,M_{mz})$ by the Wiener integrals $M_{vz} = \int_{[0,z]}\varphi_v(y)dW(y)$, where φ_v are non–random functions in $L^2([0,1]^2)$. Suppose $f(u,z)$, $u \in \mathbb{R}^m$ and $z \in [0,1]^2$, has continuous mixed partial derivatives with respect to the components of u through the third order and that

$$\tfrac{1}{2}\sum_{i,j} f^{ij}(u,z)\nabla V_{ij}(z) + \nabla f(u,z) = 0$$

where $\nabla = \mathrm{grad}_z$ and $V_{ij}(z) = \int_{[0,z]}\varphi_i(y)\varphi_j(y)dy$. Then

$$f[M(z_1,z_2),(z_1,z_2)] - f[M(z_1,0),(z_1,0)] - f[M(0,z_2),(0,z_2)] + f[M(0,0),(0,0)]$$

$$= \int_{[0,z]}\sum_i f^i(M_y,y)\varphi_i(y)dW(y)$$

$$+\tfrac{1}{2}\left[\int_{[0,z]\times[0,z]}\right]\sum_{i,j} f^{ij}(M_{y\vee y'},y\vee y')\varphi_i(y)\varphi_j(y')dW(y)dW(y').$$

<u>Remark</u> Let $\alpha:[0,1] \to [0,1]^2$ be an increasing path. For the process $\{M_{\alpha(t)}: t \in [0,1]\}$, there is the differentiation formula [13]:

$$f(M_{\alpha(t)},\alpha(t)) - f(M_{\alpha(0)},\alpha(0)) = \sum_i \int_0^t f^i(M_{\alpha(s)},\alpha(s))dM_i(\alpha(s)) +$$

$$\tfrac{1}{2}\int_0^t[\sum_{i,j} f^{ij}(M_{\alpha(s)},\alpha(s))\nabla V_{ij}(\alpha(s)) + \nabla f(M_{\alpha(s)},\alpha(s))]\cdot d\alpha(s).$$

Thus, the condition $\frac{1}{2}\sum_{i,j} f^{ij}(\mathbf{u},z)\nabla V_{ij}(z) + \nabla f(\mathbf{u},z) = 0$ in the above theorem insures that $f(M_z,z)$ is a martingale on every increasing path.

<u>Note</u> For $m = 1$ and $M(z) = \int_{[0,z]} dW(z')$, the conclusion of the theorem is $f(W(z_1,z_2)) - f(W(z_1,0)) - f(W(0,z_2)) + f(W(0,0)) =$

$$\int_{[0,z]} f'(W(\mathbf{u}))dW(\mathbf{u}) + \tfrac{1}{2}\left[\int_{[0,z]\times[0,z]}\right] f''(W(\mathbf{u}\vee\mathbf{u}'))dW(\mathbf{u})dW(\mathbf{u}').$$

Cairoli and Walsh [2] developed an Ito formula for the process $\{f(W(z)): z \in [0,1]^2\}$ where the only condition is that f is $C^3(\mathbb{R})$. Therefore, $f(W(z))$ may not be a martingale on increasing paths. The formula involves the Ito integral, a stochastic integral with respect to a process $J(z)$, path integrals, and mixed integrals.

For a fixed $t > 0$, $W(s,t)/\sqrt{t}$ is a Brownian motion, as is $W(s,t)/\sqrt{s}$ for a fixed s. The process $X(u) = W(u,t)$ for a fixed t has all the properties of a Brownian motion except that $E(X(s)X(u)) = t(s\wedge u)$. If we denote by $\partial_1 W := d_s W(s,t)$ integration along the horizontal line through $(0,t)$ with respect to $X(u)$, then the Ito formula in differential notation is

$$\partial_1 f(W(s,t),s,t) = \frac{\partial f}{\partial x}(W(s,t),s,t)\,\partial_1 W + \left[\frac{t}{2}\frac{\partial^2 f}{\partial x}(W(s,t),s,t) + \frac{\partial f}{\partial s}(W(s,t),s,t)\right]ds. \tag{1.2.1}$$

We also have the symmetric equation holding for $\partial_2 W$. Integrals with respect to $\partial_1 W$ and $\partial_2 W$ are called stochastic path integrals.

The method used in constructing the Ito integral for $d = 2$ can be applied if the Wiener process is replaced by the process

$$J(z) = \tfrac{1}{2}\left[\int_{[0,z]\times[0,z]}\right] dW(y)dW(y'). \tag{1.2.2}$$

The integral of a process φ with respect to J is denoted $\int \varphi(z)dJ(z)$.

<u>Theorem 1.2.4</u> (Green's Formula) [2]

Suppose $f \in C^2(\mathbb{R})$ with $f'(W)$ and $f''(W)$ both in $L^2(\Omega\times[0,z])$, $z \in \mathbb{R}^2_+$ and $t > 0$, fixed. For a rectangle $A \subset \mathbb{R}_z$,

$$\int_{\partial A} f(W)\partial_1 W = \int_A f(W)dW + \int_A f'(W)dJ + \iint_A \tfrac{t}{2} f''(W)\partial_1 W dt$$

where the path integral is taken in the clockwise direction.

<u>Note</u> The mixed integral is defined in the obvious way. One can replace $\partial_1 W$ with $\partial_2 W$ if the orientation is reversed.

<u>Example 1.2.1</u>

Take $f(x) = x$ and $A = \mathbb{R}_{(s,t)}$ in the above theorem to get that

$$\tfrac{1}{2}(W(s,t)^2 - st) = J(s,t) + \int_{\mathbb{R}_{(s,t)}} W(z)dW(z)$$

i.e.

$$\int_{\mathbb{R}_{(s,t)}} W(z)dW(z') = \tfrac{1}{2}(W(s,t)^2 - st) - J(s,t)$$

<u>**Theorem 1.2.5**</u> (Ito formula) [2]

For $f \in C^3(\mathbb{R})$ with $f'(W)$, $f''(W)$, $f'''(W)$ in $L^2(\Omega\times[\mathbf{0},z_0])$, and $(s,t) \in [\mathbf{0},z_0]$,

$$f(W(s,t)) - f(0) = \int_{\mathbb{R}_{(s,t)}} f'(W(z))dW(z) + \int_{\mathbb{R}_{(s,t)}} f''(W(z))dJ(z)$$

$$+ \int_0^t \left[\int_0^s \tfrac{u}{2} f'''(W(u,v))d_uW(u,v)\right]dv + \tfrac{t}{2}\int_0^s f''(W(u,t))du.$$

Chapter 2. White noise and multiparameter stochastic integration

§1. Stochastic integrals using the Hida theory.

Recall (definition 1.1.6) that $\int_{[a,b]^d} \overset{*}{\partial}_{\mathbf{u}} \varphi(\mathbf{u}) d\mathbf{u}$, if it exists, is the generalized Wiener functional with U–functional

$$U\left(\int_{[a,b]^d} \overset{*}{\partial}_{\mathbf{u}} \varphi(\mathbf{u}) d\mathbf{u}\right)(\xi) = \int_{[a,b]^d} U(\overset{*}{\partial}_{\mathbf{u}} \varphi(\mathbf{u}))(\xi) d\mathbf{u} = \int_{[a,b]^d} \xi(\mathbf{u}) U(\varphi(\mathbf{u}))(\xi) d\mathbf{u}.$$

For a non–anticipating process $\varphi(t,x)$, $t \in [a,b] \subset \mathbb{R}_+^1$ and $x \in \mathscr{S}^*(\mathbb{R})$, such that $E\int_a^b |\varphi(t)|^2 dt < \infty$, we know that

$$\int_a^b \varphi(t,x) dB(t,x) = \int_a^b \overset{*}{\partial}_t \varphi(t) dt$$

where the integral on the left is the Ito integral with respect to $B(t,x) = \langle x, 1_{[0,t[} \rangle$, see theorem 1.1.6. The following result shows that this is also true for d–dimensional time where $B(t,x)$ is replaced with the Wiener process $W(\mathbf{t},x) = \langle x, 1_{[\mathbf{0},\mathbf{t}[} \rangle$, $\mathbf{t} \in \mathbb{R}_+^d$ and $x \in \mathscr{S}^*(\mathbb{R}^d)$.

<u>Theorem 2.1.1</u>

If $\varphi(\tau,x)$, $\tau \in \mathbb{R}^d$, $x \in \mathscr{S}^*(\mathbb{R}^d)$, $d > 1$ is a nonanticipating process such that $E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau < \infty$, then

$$\int_{[a,b]^d} \varphi(\tau) dW(\tau) = \int_{[a,b]^d} \overset{*}{\partial}_\tau \varphi(\tau) d\tau$$

where the integral on the left is the d–dimensional Ito integral (see 1.2).

<u>Lemma 2.1.1</u> Let $g(u_1,u_2,\dots,u_{n+1}) = 1_{[(a,\dots,a),u_{n+1}[^n}(u_1,\dots,u_n)\, f(u_{n+1},u_1,\dots,u_n)$ where $f \in L^2(([a,b]^d)^{n+1})$ and $\hat{g}$ is the symmetrization of g. Then

$$\int_{([a,b]^d)^{n+1}} g(u_1,\dots,u_{n+1})\, dW(u_1)\dots dW(u_{n+1}) =$$

$$(n+1)\int_{[a,b]^d}\left[\int_{[(a,\dots,a),\ u_{n+1}[^n} \hat{g}(u_1,\dots,u_{n+1})\, dW(u_1)\dots dW(u_n)\right] dW(u_{n+1})$$

<u>proof.</u> Because of the cumbersome notation we prove the lemma only for $d = 2$. By ([8]) we may assume that

$$f(u_{n+1},u_1,\dots,u_n) = 1_{A_{n+1}\times A_1\times\cdots\times A_n}(u_{n+1},u_1,\dots,u_n),$$

where the A_i's are disjoint rectangles in $[a,b]^2$. Let

$$D = \{\ (x,y);\ (x,y) \in [a,b]^2 \text{ and } x \le y\ \}$$
$$D_i = [c_{i-1},c_i[,\ i = 1,\dots\ 2^m - 1, \text{ where } c_i = a + i(b-a)/2^m, \text{ and}$$
$$E_i = [c_i,b],\ i = 1,\dots,2^m - 1.$$

Then $\lim_{m\to\infty} \sum_{i=1}^{2^m-1} 1_{D_i\times E_i} = 1_D$,and since

$$1_{[(a,a),u_{n+1}[^n} = 1_{[(a,a),(s,t)]^n}((x_1,y_1),\dots,(x_n,y_n))$$
$$= 1_D(x_1,s)\ \dots\ 1_D(x_n,s)1_D(y_1,t)\ \dots 1_D(y_n,t),$$

we have that , pointwise and in $L^2([a,b]^2)^{n+1}$, $g((x_1,y_1),\dots,(x_n,y_n))$ is the limit

of $\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} \left[\prod_{p=1}^{n} 1_{D_{i_p}\times E_{i_p}}(x_p,s)1_{D_{j_p}\times E_{j_p}}(y_p,t)\right] 1_{A_1}(x_1,y_1)\dots 1_{A_{n+1}}(s,t)$, which

$$= \sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} \left[\prod_{k=1}^{n} 1_{(D_{i_k}\times D_{j_k})\cap A_k}(x_k,y_k)\right] 1_{(E_{i_1}\times E_{j_1})\cap\dots\cap(E_{i_n}\times E_{j_n})\cap A_{n+1}}(s,t).$$

Note that $(D_{i_k}\times D_{j_k}) \cap A_k < (E_{i_1}\times E_{j_1}) \cap \dots \cap (E_{i_n}\times E_{j_n}) \cap A_{n+1}$ for $k = 1,\dots,n$.

Therefore, we can see that

$$g(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) = \lim \sum_{z_{i_1},\ldots,z_{i_{n+1}}} 1_{\Delta_{z_{i_1}}}(\mathbf{u}_1) \ldots 1_{\Delta_{z_{i_n}}}(\mathbf{u}_n) 1_{\Delta_{z_{i_{n+1}}}}(\mathbf{u}_{n+1})$$

where z_{i_k} is the lower lefthand corner of $\Delta_{z_{i_k}}$, all the rectangles $\Delta_{z_{i_1}},\ldots,\Delta_{z_{i_{n+1}}}$ are disjoint, and $\Delta_{z_{i_k}} < \Delta_{z_{i_{n+1}}}$ for $k = 1,\ldots,n$. Let

$$h(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) = 1_{\Delta_{z_1}}(\mathbf{u}_1) \ldots 1_{\Delta_{z_n}}(\mathbf{u}_n) 1_{\Delta_{z_{n+1}}}(\mathbf{u}_{n+1})$$

with the above conditions. Then

$$\hat{h}(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) = \frac{1}{(n+1)!} \sum_{\pi} 1_{\Delta_{z_{\pi(1)}}}(\mathbf{u}_1) \ldots 1_{\Delta_{z_{\pi(n)}}}(\mathbf{u}_n) 1_{\Delta_{z_{\pi(n+1)}}}(\mathbf{u}_{n+1})$$

and the multiple weiner integral

$$\int_{[(a,a),\mathbf{u}_{n+1}[^n} \hat{h}(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) dW(\mathbf{u}_1) \ldots dW(\mathbf{u}_n)$$

$$= \int_{[a,b]^n} 1_{[(a,a),\mathbf{u}_{n+1}[^n}(\mathbf{u}_1,\ldots,\mathbf{u}_n) \frac{1}{(n+1)!} \sum_{\pi} \left[\prod_{k=1}^{n+1} 1_{\Delta_{z_{\pi(k)}}}(\mathbf{u}_k) \right] dW(\mathbf{u}_1)\ldots dW(\mathbf{u}_n)$$

$$= \int_{[a,b]^n} \frac{1}{(n+1)!} \left[\sum_{\pi} 1_{\Delta_{z_{\pi(1)}}}(\mathbf{u}_1) \ldots 1_{\Delta_{z_{\pi(n)}}}(\mathbf{u}_n) \right] 1_{\Delta_{z_{n+1}}}(\mathbf{u}_{n+1}) dW(\mathbf{u}_1)\ldots dW(\mathbf{u}_n)$$

$$= \frac{1}{(n+1)!} 1_{\Delta_{z_{n+1}}}(\mathbf{u}_{n+1})\, n!\, \Delta_{z_1} W \Delta_{z_2} W \ldots \Delta_{z_n} W = \frac{1}{(n+1)} 1_{\Delta_{z_{n+1}}}(\mathbf{u}_{n+1}) \Delta_{z_1} W \ldots \Delta_{z_n} W$$

is measurable with respect to $\mathscr{F}(W(\mathbf{u}_{n+1}))$ and in $L^2([a,b]^2 \times \mathscr{S}^*)$. We then have the iterated stochastic integral

$$(n+1)\int_{[a,b]^2}\left[\int_{[(a,a),u_{n+1}[^n} \hat{h}(u_1,\ldots,u_{n+1})dW(u_1) \ldots dW(u_n)\right]dW(u_{n+1}), \text{ which}$$

$$= (n+1)\int_{[a,b]^2}\left[\frac{1}{(n+1)}1_{\Delta_{z_{n+1}}}(u_{n+1})\Delta_{z_1}W \ldots \Delta_{z_n}W\right]dW(u_{n+1})$$

$$= \Delta_{z_1}W \ldots \Delta_{z_n}W\Delta_{z_{n+1}}W$$

$$= \int_{([a,b]^2)^n} h(u_1,\ldots,u_{n+1})dW(u_1,\ldots,u_{n+1}).$$

We now have that the result holds for functions of the form h and thus sums of such functions. We also know that we have a sequence of functions $\{g_m\}$ of the this form such that

$$g(u_1,\ldots,u_{n+1}) = \lim_{m\to\infty} g_m(u_1,\ldots,u_{n+1}) \text{ in } L^2(([a,b])^2)^{n+1}), \text{ and}$$
$$g(\cdot,u_{n+1}) = \lim_{m\to\infty} g_m(\cdot,u_{n+1}) \text{ in } L^2(([a,b])^2)^n.$$

Hence, in $L^2(\mathscr{S}^*)$,

$$\lim_{m\to\infty}\int_{[(a,a),u_{n+1}[^n} \hat{g}_m(u_1,\ldots,u_{n+1})dW(u_1) \ldots dW(u_{n+1})$$
$$= \int_{[(a,a),u_{n+1}[^n} \hat{g}(u_1,\ldots,u_{n+1})dW(u_1) \ldots dW(u_{n+1}), \text{ and}$$

$$\lim_{m\to\infty}\int_{[(a,a),u_{n+1}[^n} \hat{g}_m(u_1,\ldots,u_{n+1})dW(u_1) \ldots dW(u_n)$$
$$= \int_{[(a,a),u_{n+1}[^n} \hat{g}(u_1,\ldots,u_{n+1})dW(u_1) \ldots dW(u_n).$$

Notice that $\int_{[(a,a),u_{n+1}[^n} \hat{g}(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_n)$ is measurable with respect to $\mathscr{F}(W(u_{n+1}))$ and in $L^2([a,b]^2\times \mathscr{S}^*)$. Putting all of this together, we get that in $L^2(\mathscr{S}^*)$

$$\int_{([a,b]^2)^{n+1}} g(u_1,\ldots,u_{n+1})dW(u_1,\ldots,u_{n+1})$$

$$= \lim_{m\to\infty} \int_{([a,b]^2)^{n+1}} \hat{g}_m(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_{n+1})$$

$$= (n+1) \lim_{m\to\infty} \int_{[a,b]^2} \left[\int_{[(a,a),u_{n+1}[^n} \hat{g}_m(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_n)\right] dW(u_{n+1})$$

$$= (n+1)\int_{[a,b]^2} \left[\int_{[(a,a),u_{n+1}[^n} \hat{g}(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_n)\right] dW(u_{n+1})$$

since by theorem 1.2.1(b),

$$E\left|\int_{[a,b]^2} \left[\int_{[(a,a),u_{n+1}[^n} \hat{g}_m(u_1,\ldots,u_{n+1}) - \hat{g}(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_n)\right] dW(u_{n+1})\right|^2$$

$$= \int_{[a,b]^2} E\left|\int_{[(a,a),u_{n+1}[^n} \hat{g}_m(u_1,\ldots,u_{n+1}) - \hat{g}(u_1,\ldots,u_{n+1})dW(u_1)\ \ldots\ dW(u_n)\right|^2 du_{n+1}$$

$$= n!\int_{[a,b]^2} \left[\int_{[(a,a),u_{n+1}[^n} |g_m(\cdot,u_{n+1}) - g(\cdot,u_{n+1})|^2 du_1 \ldots du_n\right] du_{n+1}$$

converges to zero as $m \to \infty$.

<u>**proof of theorem.**</u> By the Wiener – Ito decomposition of $L^2(\mathscr{S}^*(\mathbb{R}^d))$, it suffices to assume that $\varphi(\tau)$ is a multiple Wiener integral. Let $T = [a,b]^d$ and suppose

$$\varphi(\tau) = \int_{(\mathbb{R}^d)^n} f(\tau;u_1,\ldots,u_n)dW(\mathbf{u}_1)\ \ldots\ dW(\mathbf{u}_n),\ \ f\in L^2(T^{n+1}).$$

Since $\varphi(\tau)$ is nonanticipating

$$\varphi(\tau) = \int_{[(a,\dots,a),\tau[^n} f(\tau;u_1,\dots,u_n)dW(u_1)\ \dots\ dW(u_n).$$

Thus $\int_T \partial_\tau^* \varphi(\tau)\, d\tau =$

$$\int_T \Big[\int_{(\mathbb{R}^d)^{n+1}} \delta_\tau(u_1)\otimes 1_{[(a,\dots,a),u_1[^n}(u_2,\dots,u_{n+1}) f(u_1,\dots,u_{n+1}) dW(u_1)\dots dW(u_{n+1})\Big]\, d\tau$$

(2.1.1) $$= \int_{(\mathbb{R}^d)^{n+1}} 1_T(u_1) 1_{[(a,\dots,a),u_1[^n}(u_2,\dots,u_n) f(u_1,\dots,u_{n+1}) dW(u_1)\dots dW(u_{n+1})$$

$$= \int_{(T)^{n+1}} 1_{[(a,\dots,a),\tau[^n}(u_1,\dots,u_n) f(\tau,u_1,\dots,u_n) dW(\tau) dW(u_1)\dots dW(u_n).$$

Let $g(u_1,\dots,u_n,\tau) = 1_{[(a,\dots,a),\tau[^n}(u_1,\dots u_n)\ f(\tau,u_1,\dots,u_n)$. Then

$$\int_T \partial_\tau^* \varphi(\tau) d\tau = \int_{(T)^{n+1}} \hat{g}(u_1,\dots,u_n,\tau)\ dW(u_1)\ \dots\ dW(u_n) dW(\tau).$$

By the lemma this equals

$$(n+1)\int_T \Big[\int_{[(a,\dots,a),\tau[^n} \hat{g}(u_1,\dots,u_n,\tau)\ dW(u_1)\ \dots\ dW(u_n)\Big] dW(\tau).$$

But for $u_1 < \tau,\dots,u_n < \tau$

$$\hat{g}(u_1,\dots,u_n,\tau) = \frac{1}{(n+1)!} \sum_{\pi_{1,\dots,n}} f(\tau,u_{\pi(1)},\dots,u_{\pi(n)})$$

$$= \frac{1}{(n+1)!}\, n!\ \hat{f}(\tau;u_1,\dots,u_n)$$ (where $\hat{f}$ is the symmetrization of f in the variables $u_1,\dots,u_n$)

$$= \frac{1}{n+1}\hat{f}(\tau;u_1,\dots,u_n).$$

Therefore,

$$\int_T \overset{*}{\partial}_\tau \varphi(\tau)d\tau = \int_T\Big[\int_{[(a,\dots,a),\tau[^n} f(\tau,u_1,\dots,u_n)dW(u_1)\quad dW(u_n)\Big]dW(\tau)$$

$$= \int_{[a,b]^d} \varphi(\tau)\, dW(\tau).$$

For $\varphi(\tau)$ not necessarily non–anticipating, the white noise integral $\int_{[a,b]^d}\overset{*}{\partial}_\tau\varphi(\tau)d\tau$, when it exists, is a natural definition of a stochastic integral of the process.

<u>Example 2.1.1</u> For $\varphi(\tau) = \int_{([a,b]^d)^n} f(\tau,u_1,\dots,u_n)dW(u_1)\dots dW(u_n)$, where $f \in \hat{L}^2(([a,b]^d)^{n+1})$, $\int_{[a,b]^d}\overset{*}{\partial}_\tau\varphi(\tau)d\tau$ exists and is given by

$$\int_{([a,b]^d)^{n+1}} f(\tau,u_1,\dots,u_n)dW(\tau)dW(u_1)\dots dW(u_n)$$

Thus,

$$E\left|\int_{[a,b]^d}\overset{*}{\partial}_\tau\varphi(\tau)d\tau\right|^2 = (n+1)!\int_{([a,b]^d)^{n+1}} |f(\tau,u_1,\dots,u_n)|^2 d\tau du_1\dots du_n$$

$$=(n+1)E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau.$$

Theorem 2.1.2

Suppose $\varphi(\tau)$ is a stochastic process such that $E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau < \infty$ and $E\int_{[a,b]^d}\int_{[a,b]^d} | \partial_{\tau'}\varphi(\tau)\, \overline{\partial_{\tau}\varphi(\tau')} \,|\, d\tau d\tau' < \infty$, then $\int_{[a,b]^d} \overset{*}{\partial}_{\tau}\varphi(\tau)d\tau$ exists and

$$E\left|\int_{[a,b]^d} \overset{*}{\partial}_{\tau}\varphi(\tau)d\tau\right|^2 = E\int_{[a,b]^d}|\varphi(\tau)|^2 d\tau + E\int_{[a,b]^2}\int_{[a,b]^2} \partial_{\tau'}\varphi(\tau)\, \overline{\partial_{\tau}\varphi(\tau')}d\tau d\tau'.$$

Note If $\varphi(\tau)$ is nonanticipating the double integral vanishes and we have the identity $E\left|\int_{[a,b]^d} \varphi(\tau)dW(\tau)\right|^2 = E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau.$

Proof. Assume that $\varphi(\tau) = \int_{(\mathbb{R}^d)^n} f(\tau;\mathbf{u}_1,\ldots,\mathbf{u}_n)dW(\mathbf{u}_1)\ldots dW(\mathbf{u}_n)$.

We may also assume that f is symmetric in the variables $\mathbf{u}_1,\ldots,\mathbf{u}_n$. Then

$$\int_{[a,b]^d} \overset{*}{\partial}_{\tau}\varphi(\tau)d\tau = \int_{(\mathbb{R}^d)^{n+1}} \hat{g}(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1})dW(\mathbf{u}_1) \ldots dW(\mathbf{u}_{n+1})$$

where $g(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) = 1_{[a,b]^d}(\mathbf{u}_1)\, f(\mathbf{u}_1;\mathbf{u}_2,\ldots,\mathbf{u}_{n+1})$. Now,

$$\hat{g}(\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}) = \frac{1}{(n+1)!}\sum_{\pi} 1_{[a,b]^d}(\mathbf{u}_{\pi(1)})\, f(\mathbf{u}_{\pi(1)};\mathbf{u}_{\pi(2)},\ldots,\mathbf{u}_{\pi(n+1)})$$

$$= \frac{1}{(n+1)!}n!\sum_{i=1}^{n+1} 1_{[a,b]^d}(\mathbf{u}_i)\, f(\mathbf{u}_i,\tilde{\mathbf{u}}_i)$$

$$= \frac{1}{(n+1)}\sum_{i=1}^{n+1} 1_{[a,b]^d}(\mathbf{u}_i)\, f(\mathbf{u}_i,\tilde{\mathbf{u}}_i)$$

where $\tilde{\mathbf{u}}_i$ denotes the variables $\mathbf{u}_1,\ldots,\mathbf{u}_{n+1}$ with $\mathbf{u}_i$ deleted. Thus

$$|\hat{g}|^2 = \frac{1}{(n+1)^2}\left[\sum_{i=1}^{n+1} 1_{[a,b]^d}(u_i)\,|f(u_i,\tilde{u}_i)|^2 + \sum_{i\neq j} 1_{[a,b]^d}(u_i)1_{[a,b]^2}(u_j)f(u_i,\tilde{u}_i)\overline{f(u_j,\tilde{u}_j)}\right].$$

Now, for any $i = 1, \ldots n+1$ we have

$$\int_{(\mathbb{R}^d)^{n+1}} 1_{[a,b]^d}(u_i)\ |f(u_i,\tilde{u}_i)|^2 du_1 \ldots du_{n+1}$$

$$= \int_{[a,b]^d}\left[\int_{(\mathbb{R}^d)^n} |f(\tau;u_1,\ldots,u_n)|^2 du_1 \ldots du_n\right]d\tau$$

$$= \int_{[a,b]^d} \frac{1}{n!}\,E|\varphi(\tau)|^2 d\tau.$$

For $i \neq j$

$$\int_{(\mathbb{R}^d)^{n+1}} 1_{[a,b]^d}(u_i)\ 1_{[a,b]^d}(u_j)\ f(u_i;\tilde{u}_i)\ \overline{f(u_j,\tilde{u}_j)}du_1 \ldots du_{n+1}$$

$$= \int_{[a,b]^d}\int_{[a,b]^d}\int_{(\mathbb{R}^d)^{n-1}} f(u_i,\tilde{u}_i)\ \overline{f(u_i,\tilde{u}_j)}du_1 \ldots du_{n+1}$$

$$= \int_{[a,b]^d}\int_{[a,b]^d}\int_{(\mathbb{R}^d)^{n-1}} f(\tau;\tau',u_2,\ldots,u_n)\overline{f(\tau';\tau,u_2,\ldots,u_n)}du_2 \ldots du_n d\tau d\tau'$$

$$= \int_{[a,b]^d}\int_{[a,b]^d} \frac{1}{(n-1)!}\frac{1}{n^2}\,E\partial_{\tau'}\varphi(\tau)\partial_{\tau}\varphi(\tau')d\tau d\tau'.$$

Hence,

$$E\left|\int_{[a,b]^d} \overset{*}{\partial}_{\tau}\varphi(\tau)d\tau\right|^2 = (n+1)!\int_{(\mathbb{R}^d)^{n+1}} |\hat{g}(u)|^2 du$$

$$= (n+1)!\frac{1}{(n+1)^2}\left[(n+1)\frac{1}{n!}\,E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau\right.$$

$$+ (n+1)n\int_{[a,b]^d}\int_{[a,b]^d} \frac{1}{(n-1)!}\frac{1}{n^2} E\partial_{\tau'}\varphi(\tau)\partial_{\tau}\varphi(\tau')d\tau d\tau'\Big]$$

$$= E\int_{[a,b]^d}|\varphi(\tau)|^2 d\tau + \int_{[a,b]^d}\int_{[a,b]^d} E\partial_{\tau'}\varphi(\tau)\overline{\partial_{\tau}\varphi(\tau')}d\tau d\tau'.$$

Theorem 2.1.3

Suppose $\varphi(\tau)$ is a stochastic process such that $E\int_{[a,b]^d}|\varphi(\tau)|^2 d\tau < \infty$ and $E\int_{[a,b]^d}\int_{[a,b]^d}|\partial_{\alpha}\varphi(\tau)|^2 d\tau d\alpha < \infty$. Then $\int_{[a,b]^d}\overset{*}{\partial}_{\tau}\varphi(\tau)d\tau$ exists and

$$\left|\int_{[a,b]^d}\overset{*}{\partial}_{\tau}\varphi(\tau)d\tau\right|^2 \le E\int_{[a,b]^d}|\varphi(\tau)|^2 d\tau + E\int_{[a,b]^d}\int_{[a,b]^d}|\partial_{\alpha}\varphi(\tau)|^2 d\tau d\alpha < \infty$$

Proof.

$$E\int_{[a,b]^d}\int_{[a,b]^d}|\partial_{\alpha}\varphi(\tau)\overline{\partial_{\tau}\varphi(\alpha)}|d\tau d\alpha$$

$$\le E\int_{[a,b]^d}\int_{[a,b]^d}\frac{1}{2}\Big[|\partial_{\alpha}\varphi(\tau)|^2 + |\partial_{\tau}\varphi(\alpha)|^2\Big]d\tau d\alpha$$

$$= E\int_{[a,b]^d}\int_{[a,b]^d}|\partial_{\tau}\varphi(\alpha)|^2 d\tau d\alpha.$$

The conclusion now follows from theorem 2.1.2.

Theorem 2.1.4

Let $\varphi(\tau)$ be a stochastic process such that $\int_{[a,b]^d}|E\varphi(\tau)|^2 d\tau < \infty$ and $E\int_{\mathbb{R}^d}\int_{[a,b]^d}|\partial_{\alpha}\varphi(\tau)|^2 d\tau d\alpha < \infty$. Then $\int_{[a,b]^d}\overset{*}{\partial}_{\tau}\varphi(\tau)d\tau$ exists and

$$E\left|\int_{[a,b]^d}\overset{*}{\partial}_{\tau}\varphi(\tau)d\tau\right|^2 \le E\int_{[a,b]^d}|\varphi(\tau)|^2 d\tau + E\int_{\mathbb{R}^d}\int_{[a,b]^d}|\partial_{\alpha}\varphi(\tau)|^2 d\tau d\alpha.$$

<u>**Lemma 2.1.4**</u> For $\varphi = \int_{(\mathbb{R}^d)^n} f(\mathbf{u}_1,\ldots,\mathbf{u}_n)dW(\mathbf{u}_1)\ldots dW(\mathbf{u}_n)$, $f \in \hat{L}^2((\mathbb{R}^d)^n)$,

$$\int_{(\mathbb{R}^d)} E|\partial_{\boldsymbol{\tau}}\varphi|^2 d\boldsymbol{\tau} = nE|\varphi|^2.$$

<u>proof</u> For almost all $\boldsymbol{\tau}$, $\partial_{\boldsymbol{\tau}}\varphi = n\int_{(\mathbb{R}^d)^{n-1}} f(\boldsymbol{\tau},\mathbf{u}_1,\ldots,\mathbf{u}_{n-1})dW(\mathbf{u}_1)\ldots dW(\mathbf{u}_{n-1})$. Therefore we have

$$E|\partial_{\boldsymbol{\tau}}\varphi|^2 = n^2(n-1)!\int_{(\mathbb{R}^d)^{n-1}} |f(\boldsymbol{\tau},\mathbf{u}_1,\ldots,\mathbf{u}_{n-1})|^2 d\mathbf{u}_1\ldots d\mathbf{u}_{n-1}.$$

Hence, $\int_{(\mathbb{R}^d)} E|\partial_{\boldsymbol{\tau}}\varphi|^2 d\boldsymbol{\tau} = n\cdot n!\int_{(\mathbb{R}^d)^n} |f(\mathbf{u}_1,\ldots,\mathbf{u}_n)d\mathbf{u}_1\ldots d\mathbf{u}_n = nE|\varphi|^2$.

<u>Note</u> Suppose $\varphi(\boldsymbol{\tau})$ is a multiple Wiener integral of order n for any $\boldsymbol{\tau} \in [a,b]^d$ such that $E\int_{[a,b]^d} |\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} < \infty$. Then

$$E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} = nE\int_{[a,b]^d} |\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau}. \tag{2.1.3}$$

<u>proof of theorem</u> Let $\varphi(\boldsymbol{\tau}) = \sum_{n=0}^{\infty} \varphi_n(\boldsymbol{\tau})$ be the expansion of $\varphi(\boldsymbol{\tau})$ into multiple Wiener integrals. By (2.1.3) above, we have for $n \geq 1$

$$E\int_{[a,b]^d} |\varphi_n(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} = \frac{1}{n}E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi_n(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} \leq E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi_n(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha}$$

Therefore,

$$\begin{aligned}
E\int_{[a,b]^d} |\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} &= E\int_{[a,b]^d} |\varphi_0(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} + \sum_{n=1}^{\infty} E\int_{[a,b]^d} |\varphi_n(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} \\
&\leq \int_{[a,b]^d} |E\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} + \sum_{n=1}^{\infty} E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi_n(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} \\
&= \int_{[a,b]^d} |E\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} + E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} \\
&< \infty.
\end{aligned}$$

On the other hand, we have

$$E\int_{[a,b]^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} \leq E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\boldsymbol{\tau})|^2 d\boldsymbol{\tau} d\boldsymbol{\alpha} < \infty$$

and the theorem follows from theorem 2.1.3.

Corollary 2.1.4 Suppose $\varphi(\tau) = \sum_{n=0}^{\infty} \varphi_n(\tau)$, $\varphi_n(\tau) \in K_n$, and $\sum_{n=0}^{\infty}(n+1)E\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau < \infty$. Then $\int_{[a,b]^d} \overset{*}{\partial}_\tau \varphi(\tau) d\tau$ exists and

$$E\left|\int_{[a,b]^d} \overset{*}{\partial}_\tau \varphi(\tau) d\tau\right|^2 \le \sum_{n=0}^{\infty}(n+1)E\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau.$$

Remark In view of example 2.1.1, this inequality cannot be improved.

proof By (2.1.3) we have

$$E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\tau)|^2 d\tau d\boldsymbol{\alpha} = \sum_{n=1}^{\infty} nE\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau < \infty.$$

On the other hand, $\int_{[a,b]^d} |E\varphi(\tau)|^2 d\tau = \int_{[a,b]^d} |\varphi_0(\tau)|^2 d\tau < \infty$.
Therefore we have by theorem 2.1.4, that the integral $\int_{[a,b]^d} \overset{*}{\partial}_\tau \varphi(\tau) d\tau$ exists and

$$\begin{aligned}
E\left|\int_{[a,b]^d} \overset{*}{\partial}_\tau \varphi(\tau) d\tau\right|^2 &\le E\int_{[a,b]^d} |\varphi(\tau)|^2 d\tau + E\int_{\mathbb{R}^d}\int_{[a,b]^d} |\partial_{\boldsymbol{\alpha}}\varphi(\tau)|^2 d\tau d\boldsymbol{\alpha} \\
&= \sum_{n=0}^{\infty} E\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau + \sum_{n=1}^{\infty} nE\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau \\
&= \sum_{n=0}^{\infty}(n+1)E\int_{[a,b]^d} |\varphi_n(\tau)|^2 d\tau.
\end{aligned}$$

We will next consider the integral defined by Wong and Zakai [20], see 1.2. This integral requires measurability with respect to $\mathcal{F}_{z\vee z'}$, $z,z' \in [a,b]^2$. We show that it also can be expressed as a white noise integral.

<u>Theorem 2.1.5</u>

Given a jointly measurable function $\varphi(x,z,z')$ on $\mathscr{S}^*(\mathbb{R}^2)\times[a,b]^2\times[a,b]^2$ such that

(1) For each pair z,z', $\varphi(x,z,z')$ is measurable with respect to $\mathscr{F}_{z\vee z'}$

(2) $E\int_{[a,b]^2}\int_{[a,b]^2} \varphi^2(z,z')dzdz' < \infty$,

it follows that

$$\int_{[a,b]^2}\int_{[a,b]^2} \partial^*_{z'}\partial^*_{z}\, 1_G(z,z')\varphi(z,z')dzdz' = \left[\int_{[a,b]^2\times[a,b]^2}\right] \varphi(z,z')dW(z)dW(z').$$

Here $G = \{ (z,z') \in [a,b]^2 \times [a,b]^2$ such that z and z' are unordered $\}$

and $\left[\int_{[a,b]^2\times[a,b]^2}\right]$ denotes the Wong and Zakai integral.

<u>Lemma 2.1.5</u> Let $T = [a,b]^2$. For $f \in L^2(T)^{n+2}$, let

$$g(u_1,\dots,u_n,z,z') = 1_{[(a,a),z\vee z'[^n}(u_1,\dots,u_n)\, 1_G(z,z')f(z,z',u_1,\dots u_n).$$

Then, $\int_T \cdots \int_T \hat{g}(u_1,\dots u_n,z,z')dW(u_1)\ \dots\ dW(u_n)dW(z)dW(z') =$

$$(n+2)(n+1)\left[\int_{T\times T}\right]\left[\int_{[(a,a),z\vee z'[^n}\hat{g}(u_1,\dots,u_n,z,z')dW(u_1)\dots dW(u_n)\right]dW(z)dW(z')$$

<u>proof</u>. As in lemma 2.1.1 we may assume that

$$f(z,z',u_1,\dots,u_n) = 1_{A_{n+2}\times A_{n+1}\times A_1\times \dots A_n}(z,z',u_1,\dots u_n).$$

where the A_i's are disjoint rectangles in $[a,b]^2$.

Notice that $G = G_1 \cup G_2$, where $G_1 = \{((s,t),(s_1,t_1)) \mid s < s_1 \text{ and } t > t_1\}$ and $G_2 = \{((s,t),(s_1,t_1)) \mid s > s_1 \text{ and } t < t_1\}$. Let $D = \{(x,s) \mid x < s\}$ and $D' = \{(x,s) \mid x > s\}$. Then

$1_{G_1}(s,t,s_1,t_1) = 1_D(s,s_1)1_{D'}(t,t_1)$ and $1_{G_2}(s,t,s_1t_1) = 1_{D'}(s,s_1)1_D(t,t_1)$.

Let $P_q = \{c_i\}$ be the partition of $[a,b]$ in segments of length $(b-a)/2^q$, $D_i = [c_{i-1},c_i]$, $E_i = [c_i,b]$, $E_i' = [a,c_i[$, so that

$$1_D(s,s_1) = \lim_{q\to\infty} \sum 1_{D_i\times E_i}(s,s_1) \text{ and } 1_{D'}(t,t_1) = \lim_{n\to\infty} \sum 1_{D_i\times E_i'}(t,t_1).$$

Thus

$$1_{[(a,a),(s,t)\vee(s_1,t_1)[^n}((x_1,y_1),\ldots,(x_n,y_n)1_G((s,t),(s_1,t_1))$$

$$= 1_{[(a,a),(s,t)\vee(s_1,t_1)[^n}((x_1,y_1),\ldots,(x_n,y_n)1_{G_1}((s,t),(s_1,t_1))$$
$$+ 1_{[(a,a),(s,t)\vee(s_1,t_1)[^n}((x_1,y_1),\ldots,(x_n,y_n)1_{G_2}((s,t),(s_1,t_1))$$

$$= 1_{[(a,a),(s_1,t)[^n}((x_1,y_1),\ldots,(x_n,y_n)1_{G_1}((s,t),(s_1,t_1))$$
$$+ 1_{[(a,a),(s,t_1)[^n}((x_1,y_1),\ldots,(x_n,y_n)1_{G_2}((s,t),(s_1,t_1))$$

$$= 1_D(x_1,s_1) \ldots 1_D(x_n,s_1)1_D(y_1,t) \ldots 1_D(y_n,t)1_D(s,s_1)1_{D'}(t,t_1)$$
$$+ 1_D(x_1,s_1) \ldots 1_D(x_n,y_n)1_D(y_1,t) \ldots 1_D(y_n,t)1_{D'}(s,s_1)1_D(t,t_1)$$

giving that $g((x_1,y_1),\ldots,(x_n,y_n),(s,t),(s_1,t_1))$ is the limit of

$$\sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n\\ k,m}} \Big[\prod_{p=1}^{n} 1_{D_{i_p}\times E_{i_p}}(x_p,s_1)1_{D_{j_p}\times D_{j_p}}(y_p,t)1_{D_k\times E_k}(s,s_1)1_{D_m\times E_m'}(t,t_1) +$$

$$\prod_{p=1}^{n} 1_{D_{i_p}\times E_{i_p}}(x_p,s)1_{D_{j_p}\times E_{j_p}}(y_p,t_1)1_{D_k\times E_k'}(s,s_1)1_{D_m\times E_m}(t,t_1)\Big]1_{A_{n+2}\times A_{n+1}\times A_1\times\ldots A_n}((s,t),(s_1,t_1),(x_1,y_1),\ldots,(x_n,y_n))$$

$$= \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_n \\ k,m}} \prod_{p=1}^{n} 1_{D_{i_p}\times D_{j_p}\times A_p}(x_n,y_n) \Big[1_{(D_k\times(\cap E_{j_i}\cap D_m))\cap A_{n+2}}(s,t)\, 1_{((\cap E_{i_j}\cap E_k)\times E_m')\cap A_{n+1}}(s_1,t_1)$$

$$+ 1_{((\cap E_{i_j}\cap D_k)\times D_m)\cap A_{n+2}}(s,t)\, 1_{(E_k'\times(\cap E_{j_i}\cap E_m))\cap A_{n+1}}(s_1,t_1)\Big].$$

Upon inspection one sees that

$$g(u_1,\dots,u_n,z,z') = \lim \sum_{i_1,\dots i_{n+2}} 1_{\Delta_{z_{i_1}}}(u_1) \dots 1_{\Delta_{z_n}}(u_n) 1_{\Delta_{z_{i_{n+1}}}}(z) 1_{\Delta_{z_{i_{n+2}}}}(z')$$

where $\Delta_{z_{i_1}},\dots,\Delta_{z_{i_{n+2}}}$ are disjoint rectangles , $\Delta_{z_{i_{n+1}}}\times\Delta_{z_{i_{n+2}}} \subset G$, and $\Delta_{z_{i_{n+1}}} \leq \Delta_{z_{i_{n+1}}\vee z_{i_{n+2}}}$ for k = 1,...,n. Consider

$$h(u_1,\dots,u_n,z,z') = 1_{\Delta_{z_1}}(u_1) \dots 1_{\Delta_{z_n}}(u_n) 1_{\Delta_{z_{n+1}}}(z) 1_{\Delta_{z_{n+2}}}(z')$$

with the above conditions We obtain that

$$\int_{[(a,a),z\vee z'[^n} \hat{h}(u_1,\dots,u_n,z,z')dW(u_1) \dots dW(u_n)$$

$$= \int_{T^n} 1_{[(a,a),z\vee z'[^n}(u_1,\dots,u_n) \frac{1}{(n+2)!} \sum_{\pi} 1_{z_{\pi(1)}}(u_1) \dots 1_{z_{\pi(n+1)}}(z)\, 1_{z_{\pi(n+2)}}(z')dW(u_1)\dots dW(u_n)$$

$$= \int_{T^n} \frac{1}{(n+2)!} \sum_{\tau} 1_{\Delta_{z_{\tau(1)}}}(u_1) \dots 1_{\Delta_{z_{\tau(n)}}}(u_n)\, 1_{\Delta_{z_{n+1}}}(z) 1_{\Delta_{z_{n+2}}}(z')$$

$$= \frac{1}{(n+2)!} n!\, 1_{\Delta_{z_{n+1}}}(z)\, 1_{\Delta_{z_{n+2}}}(z') \Delta_{z_1}W \dots \Delta_{z_n}W$$

$$= \frac{1}{(n+2)(n+1)} 1_{\Delta_{z_{n+1}}}(z)\, 1_{\Delta_{z_{n+2}}}(z') \Delta_{z_1}W \dots \Delta_{z_n}W$$

which is measurable with respect to $\mathscr{F}(W(z \vee z'))$ and in $L^2(T \times T \times \mathscr{S}^*)$.

We can then look at the Wong and Zakai integral:

$$(n+2)(n+1)\left[\int_{T\times T}\right]\left[\int_{[(a,a),z\vee z'[^n} \hat{h}(u_1,\ldots,u_n,z,z')dW(u_1)\ldots dW(u_n)\right]dW(z)dW(z')$$

$$= \left[\int_{T\times T}\right]\left[1_{\Delta_{z_{n+1}}}(z)1_{\Delta_{z_{n+2}}}(z')\ \Delta_{z_1}W \ldots \Delta_{z_n}W\right]dW(z)dW(z')$$

$$= \Delta_{z_1}W \ldots \Delta_{z_n}W\ \Delta_{z_{n+1}}W\ \Delta_{z_{n+2}}W$$

$$= \int_{T^{n+2}} h(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)dW(z)dW(z').$$

Now, $g(u_1,\ldots,u_n,z,z') = \lim_{m\to\infty} g_m(u_1,\ldots,u_n,z,z')$ where $\{g_m\}$ is a sequence of sums of functions of the above form. We have that in $L^2(\mathscr{S}^*)$

$$\int_{T^{n+2}} \hat{g}(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)dW(z)dW(z')$$

$$= \lim_{m\to\infty} \int_{T^{n+2}} \hat{g}_m(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)dW(z)dW(z')$$

$$= (n+2)(n+1)\lim_{m\to\infty}\left[\int_{T\times T}\right]\left[\int_{[(a,a),z\vee z'[^n} \hat{g}_m(u_1,..,u_n,z,z')dW(u_1)..dW(u_n)\right] \times dW(z)dW(z')$$

$$= (n+2)(n+1)\left[\int_{T\times T}\right]\left[\int_{[(a,a),z\vee z'[^n} \hat{g}(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)\right]dW(z)dW(z')$$

since (see theorem 1.2.2(c))

$$E\left|\left[\int_{T\times T}\right]\left[\int_{[(a,a),z\vee z'[^n}(\hat{g}_m-\hat{g})(u_1,\dots,u_n,z,z')dW(u_1)\dots dW(u_n)\right]dW(z)dW(z')\right|^2$$

$$= E\int_{T^2}1_G(z,z')\left|\int_{[(a,a),z\vee z'[^n}(\hat{g}_m-\hat{g})(u_1,\dots,u_n,z,z')dW(u_1)\dots dW(u_n)\right|^2 dzdz'$$

$$= \int_{T^2}1_G(z,z')E\left|\int_{[(a,a),z\vee z'[^n}(\hat{g}_m-\hat{g})(u_1,\dots,u_n,z,z')dW(u_1)\dots dW(u_n)\right|^2 dzdz'$$

$$= \int_{T^2}1_G(z,z')n!\left[\int_{[(a,a),z\vee z'[^n}|(\hat{g}_m-\hat{g})(u_1,\dots,u_n,z,z')|^2 du_1\dots du_n\right]dzdz'$$

converges to zero as $m \to \infty$.

<u>proof of theorem</u> Here again let $T = [a,b]^2$. Also, set $[(a,a),z\vee z'[= D$. Assume

$$\varphi(x,z,z') = \int_D\cdots\int_D f(z,z',u_1,\dots,u_n)dW(u_1)\dots dW(u_n),\ f \in L^2(T^{n+2}).$$

Then,

$$1_G(z,z')\varphi(z,z',x) = \int_D\cdots\int_D 1_G(z,z')f(z,z',u_1,\dots,u_n)dW(u_1),\dots dW(u_n) \text{ and,}$$

$$\int_T\int_T \overset{*}{\partial}_{z'}\overset{*}{\partial}_z 1_G(z,z')\varphi(z,z')dzdz' =$$

$$\int_{T^2}\int_{(\mathbb{R}^2)^{n+2}}\delta_z(u_1)\otimes\delta_{z'}(u_2)\otimes 1_{D^n}(u_3,\dots,u_{n+2})$$
$$\times\ 1_G(u_1,u_2)f(u_1,\dots,u_{n+2})dW(u_1)\dots dW(u_{n+2})dzdz'$$

$$= \int_{T^{n+2}}1_{D^n}(u_1,\dots,u_n)1_G(z,z')f(z,z',u_1,\dots,u_n)dW(u_1)\dots dW(u_n)dW(z)dW(z').$$

Let $g(u_1,\dots u_n,z,z') = 1_D(u_1,\dots,u_n)1_G(z,z')f(z,z',u_1,\dots u_n)$. Then

$$\int_{T^2}\int_{T^2} \overset{*}{\partial}_{z'} \overset{*}{\partial}_{z} 1_G(z,z')\varphi(z,z')dzdz'$$

$$= \int_{(T^2)^{n+2}} \hat{g}(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)dW(z)dW(z')$$

$$= (n+2)(n+1)\left[\int_{T\times T}\right]\left[\int_{D^n} \hat{g}(u_1,\ldots,u_n,z,z')dW(u_1) \ldots dW(u_n)\right]dW(z)dW(z').$$

Note that for $u_1,\ldots,u_n < z \vee z'$

$$\hat{g}(u_1,\ldots,u_n,z,z') = \frac{1}{(n+2)(n+1)} 1_G(z,z')\hat{f}(z,z',u_1,\ldots,u_n)$$

so that $\int_T\int_T \overset{*}{\partial}_{z'} \overset{*}{\partial}_{z} 1_G(z,z')\varphi(z,z')dzdz'$

$$= \left[\int_{T\times T}\right]\left[\int_{D^n} 1_G(z,z')f(z,z',u_1,\ldots u_n)dW(u_1) \ldots dW(u_n)\right]dW(z)dW(z')$$

$$= \left[\int_{T\times T}\right](1_G(z,z')\varphi(z,z'))dW(z)dW(z')$$

$$= \left[\int_{T\times T}\right]\varphi(z,z')dW(z)dW(z').$$

We now consider the existence of $\int_{(([a,b]^2)^2} \overset{*}{\partial}_{z'} \overset{*}{\partial}_{z}\varphi(z,z')dzdz'$ where $\varphi(z,z')$ may not be measurable with respect to $\mathscr{F}_{z\vee z'}$.

__Theorem 2.1.6__ Let $T = [a,b]^2$ and $\varphi(z,z')$ be a stochastic process such that

$$\int_{T^2} E(|\varphi(z,z')|^2 + |\varphi(z,z')\overline{\varphi(z',z)}|)dzdz' < \infty,$$

$$\int_{T^3} E|\partial_v\varphi(z,z')(\overline{\partial_{z'}\varphi(z,v)} + \overline{\partial_{z'}\varphi(v,z)} + \overline{\partial_z\varphi(v,z')} + \overline{\partial_z\varphi(z',v)})|dvdzdz' < \infty,$$

and $\int_{T^4} E|\partial_u\partial_v\varphi(z,z')\ \overline{\partial_z\partial_{z'}\varphi(u,v)}|dudvdzdz' < \infty.$

Then $\int_{T^2} \overset{*}{\partial}_{z'}\overset{*}{\partial}_z\varphi(z,z')dzdz'$ exists and $E\left|\int_{T^2}\overset{*}{\partial}_{z'}\overset{*}{\partial}_z\varphi(z,z')dzdz'\right|^2 =$

$$\int_{T^2} E(|\varphi(z,z')|^2 + \varphi(z,z')\overline{\varphi(z',z)})dzdz' +$$

$$\int_{T^3} E[\partial_v\varphi(z,z')(\overline{\partial_{z'}\varphi(z,v)} + \overline{\partial_{z'}\varphi(v,z)} + \overline{\partial_z\varphi(v,z')} + \overline{\partial_z\varphi(z',v)})]dvdzdz' +$$

$$\int_{T^4} E\ \partial_u\partial_v\varphi(z,z')\overline{\partial_z\partial_{z'}\varphi(u,v)}dudvdzdz'.$$

__proof__. Assume that $\varphi(z,z') = \int_{(\mathbb{R}^2)^n} f(z,z',u_1,\ldots,u_n)dW(u_1)\ \ldots\ dW(u_n)$ where $f(z,z',\cdot) \in \hat{L}^2(T^n)$ and $f \in L^2(T^{n+2})$. Then

$$\int_{T^2}\overset{*}{\partial}_{z'}\overset{*}{\partial}_z\varphi(z,z')dzdz' = \int_{(\mathbb{R}^2)^{n+2}} 1_T(u_1)1_T(u_2)f(u_1,\ldots,u_{n+2})dW(u_1)\ \ldots\ dW(u_{n+2}).$$

Let $g(u_1,\ldots,u_{n+2}) = 1_T(u_1)1_T(u_2)f(u_1,\ldots,u_{n+2})$ so that

$$\hat{g}(u_1, \quad ,u_{n+2}) = \frac{1}{(n+2)!}\sum_\pi 1_T(u_{\pi(1)})1_T(u_{\pi(2)})f(u_{\pi(1)},\ldots,u_{\pi(n+2)})$$

$$= \frac{1}{(n+2)!} n! \sum_{i=1}^{n+2} \sum_{i \neq j} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}).$$

Here $\tilde{u}_{ij}$ denotes the variables $u_1, \ldots, u_{n+2}$ with u_i and u_j deleted. Thus

$$|\hat{g}|^2 = \frac{1}{(n+2)^2(n+1)^2} \left| \sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1(u_j) f(u_i, u_j, \tilde{u}_{ij}) \right|^2$$

$$= \frac{1}{(n+2)^2(n+1)^2} \Bigg[\sum_{i=1}^{n+2} \left| \sum_{j \neq i} 1_T(u_i) 1(u_j) f(u_i, u_j, \tilde{u}_{ij}) \right|^2 +$$

$$+ \sum_{i=m} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) \sum_{k \neq m} 1_T(u_m) 1_T(u_k) \overline{f(u_l, u_k, \tilde{u}_{mk})} \Bigg]$$

$$= \frac{1}{(n+2)^2(n+1)^2} \Bigg[\sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) |f(u_i, u_j, \tilde{u}_{ij})|^2$$

$$+ \sum_{i=1}^{n+2} \sum_{\substack{k \neq i \\ j \neq k,\, i}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_i) 1_T(u_k) \overline{f(u_i, u_k, \tilde{u}_{ik})}$$

$$+ \sum_{i \neq m} \sum_{\substack{j \neq i \\ k \neq m}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_m) 1_T(u_k) \overline{f(u_m, u_k, \tilde{u}_{mk})} \Bigg]$$

$$= \frac{1}{(n+2)^2(n+1)^2} \Bigg[\sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) |f(u_i, u_j, \tilde{u}_{ij})|^2$$

$$+ \sum_{i=1}^{n+2} \sum_{\substack{j,k \neq i \\ j \neq k}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_k) \overline{f(u_i, u_k, \tilde{u}_{ik})}$$

$$= \frac{1}{(n+2)!} n! \sum_{i=1}^{n+2} \sum_{i \neq j} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}).$$

Here $\tilde{u}_{ij}$ denotes the variables $u_1, \ldots, u_{n+2}$.with u_i and u_j deleted. Thus

$$|\hat{g}|^2 = \frac{1}{(n+2)^2(n+1)^2} \left| \sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1(u_j) f(u_i, u_j, \tilde{u}_{ij}) \right|^2$$

$$= \frac{1}{(n+2)^2(n+1)^2} \left[\sum_{i=1}^{n+2} \left| \sum_{j \neq i} 1_T(u_i) 1(u_j) f(u_i, u_j, \tilde{u}_{ij}) \right|^2 + \right.$$

$$\left. + \sum_{i=m} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) \sum_{k \neq m} 1_T(u_m) 1_T(u_k) \overline{f(u_l, u_k, \tilde{u}_{mk})} \right]$$

$$= \frac{1}{(n+2)^2(n+1)^2} \left[\sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) |f(u_i, u_j, \tilde{u}_{ij})|^2 \right.$$

$$+ \sum_{i=1}^{n+2} \sum_{\substack{k \neq i \\ j \neq k,\, i}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_i) 1_T(u_k) \overline{f(u_i, u_k, \tilde{u}_{ik})}$$

$$\left. + \sum_{i \neq m} \sum_{\substack{j \neq i \\ k \neq m}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_m) 1_T(u_k) \overline{f(u_m, u_k, \tilde{u}_{mk})} \right]$$

$$= \frac{1}{(n+2)^2(n+1)^2} \left[\sum_{i=1}^{n+2} \sum_{j \neq i} 1_T(u_i) 1_T(u_j) |f(u_i, u_j, \tilde{u}_{ij})|^2 \right.$$

$$+ \sum_{i=1}^{n+2} \sum_{\substack{j,k \neq i \\ j \neq k}} 1_T(u_i) 1_T(u_j) f(u_i, u_j, \tilde{u}_{ij}) 1_T(u_k) \overline{f(u_i, u_k, \tilde{u}_{ik})}$$

$$+\sum_{i\neq m}\ \sum_{j\neq i,m} 1_T(u_i)1_T(u_j)1_T(u_m)f(u_i,u_j,\tilde{u}_{ij})\overline{f(u_m,u_j,\tilde{u}_{mj})}$$

$$+\sum_{i\neq m}\ \sum_{\substack{k\neq m,i\\ j\neq k,i,m}} 1_T(u_i)1_T(u_j)1_T(u_m)1_T(u_k)f(u_i,u_j,\tilde{u}_{ij})\overline{f(u_m,u_k,\tilde{u}_{mk})}$$

$$+\sum_{i\neq m}\ \sum_{k\neq m,i} 1_T(u_i)1_T(u_m)1_T(u_k)f(u_i,u_m,\tilde{u}_{im})\overline{f(u_m,u_k,\tilde{u}_{mk})}$$

$$+\sum_{i\neq m}\ \sum_{\substack{m\neq j\\ j\neq i}} 1_T(u_i)1_T(u_j)1_T(u_m)f(u_i,u_j,\tilde{u}_{ij})\overline{f(u_m,u_i,\tilde{u}_{mi})}$$

$$+\sum_{i\neq m} 1_T(u_i)1_T(u_m)f(u_i,u_m,\tilde{u}_{im})\overline{f(u_m,u_i,\tilde{u}_{mi})}\Big].$$ Thus,

$$E\left|\int_{T^2}\overset{*}{\partial}_{z'}\overset{*}{\partial}_{z}\varphi(z,z')dzdz'\right|^2 = (n+2)!\int_{(\mathbb{R}^2)^{n+2}}|\hat{g}(u_1,\dots,u_{n+2})|^2du_1\ \dots\ du_{n+2}$$

$$= \frac{(n+2)!}{(n+2)^2(n+1)^2}\Big[(n+2)(n+1)\frac{1}{n!}\int_{T^2}E|\varphi(z,z')|^2dzdz'$$

$$+\ (n+2)(n+1)n\frac{1}{(n-1)!}\frac{1}{n^2}\int_{T^3}E\partial_v\varphi(z,z')\overline{\partial_{z'}\varphi(z,v)}dvdzdz'$$

$$+\ (n+2)(n+1)n\frac{1}{(n-1)!}\frac{1}{n^2}\int_{T^3}E\partial_v\varphi(z,z')\overline{\partial_{z}\varphi(v,z')}dvdzdz'$$

$$+\ (n+2)(n+1)n\frac{1}{(n-1)!}\frac{1}{n^2}\int_{T^3}E\partial_v\varphi(z,z')\overline{\partial_{z}\varphi(z',v)}dvdzdz'$$

$$+ (n+2)(n+1)n\frac{1}{(n-1)!}\frac{1}{n^2}\int_{T^3} E\partial_v\varphi(z,z')\overline{\partial_{z'}\varphi(v,z)}dvdzdz'$$

$$+(n+2)(n+1)(n)(n-1)\frac{1}{n^2}\frac{1}{(n-1)^2}\frac{1}{(n-2)!}\int_{T^4} E\partial_u\partial_v\varphi(z,z')\overline{\partial_z\partial_{z'}\varphi(u,v)}\ dudvdzdz'$$

$$+ (n+2)(n+1)\frac{1}{n!}\int_{T^2} E\varphi(z,z')\overline{\varphi(z',z)}dzdz'\Big]$$, and the result

follows.

§ 2. A generalized Ito formula for 2–dimensional time.

In order to develop our Ito formula, we will need the following result. This theorem defines the generalized Wiener functional $F(W(s,t))$ for $F \in \mathscr{S}^*(\mathbb{R})$ and gives its S–transform. This is the analogue of the one dimensional time case given by Kuo [16] and Kubo (theorem 1.1.7).

<u>Theorem 2.2.1</u> For $t,s > 0$, and $H_n(x) = (-1)^n\exp(x^2)D_x^n \exp(-x^2)$

(a) $\delta_x(W(t,s)) := (2\pi ts)^{-\frac{1}{2}} \exp(-x^2/2ts)\sum_{n=0}^{\infty} (n!2^n)^{-1}H_n(x/\sqrt{2ts})H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})$,

where $\rho(ts)(u) = 1_{(0,(t,s)]}(u)/\sqrt{ts}$, is in $(L^2)^-$,

(b) For $F \in \mathscr{S}^*(\mathbb{R})$, $F(W(t,s)) :=$

$$\sum_{n=0}^{\infty} (2\pi ts)^{-\frac{1}{2}}(n!2^n)^{-1}H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\int_{-\infty}^{\infty} \exp(-x^2/(2ts)H_n(x/\sqrt{2ts})F(x)dx$$

is in $(L^2)^-$, and

(c) $S(F(W(t,s))(\xi) = g_{ts} * F\langle\xi,1_{(0,(t,s)]}\rangle$, where $g_u(y) = (2\pi u)^{-\frac{1}{2}}\exp(-y^2/2u)$.

<u>Lemma 2.2.1(a)</u> For $f \in L^2(\mathbb{R}^2)$ with $\|f\| = 1$, $H_n(\langle \cdot, f\rangle/\sqrt{2}) \in K_n^{(-n)}$ with

$$\| H_n(\langle \cdot, f\rangle/\sqrt{2})\|^2_{K_n^{(-n)}} = n!2^n \int_{\mathbb{R}^{2n}} (1+|s|^2)^{-\left(\frac{2n+1}{2}\right)} |\hat{f}(s_1) \dots \hat{f}(s_n)|^2 ds_1 \dots ds_n$$

where $\hat{f}$ is the Fourier transform of f.

<u>proof</u>. Recall that $H_n(\langle \cdot, f\rangle/\sqrt{2})$ is represented by $F_n(s_1,\dots,s_n)$ $= (\sqrt{2})^n f(s_1) \dots f(s_n)$. Thus $\hat{F}_n(s_1,\dots,s_n) = (\sqrt{2})^n \hat{f}(s_1) \dots \hat{f}(s_n)$. By [21, p155],

$$\|F_n\|^2_{\hat{H}^{\frac{-2n+1}{2}}[(\mathbb{R}^2)^n]} = \int_{\mathbb{R}^{2n}} (1+|s|^2)^{-\left(\frac{2n+1}{2}\right)} |\hat{F}_n(s)|^2 ds.$$ The lemma then follows

by the definition of $K_n^{(-n)}$.

<u>Lemma 2.2.1(b)</u> For $n \geq 1$

$$\| H_n(\langle \cdot, \rho(ts)\rangle/\sqrt{2})\|^2_{K_n^{(-n)}} \leq n!2^{n+1}(ts/4\pi^2)^n \sigma_{2n}$$

where σ_{2n} is the surface area of the unit sphere in $\mathbb{R}^{2n}$.

<u>proof</u>.

$$\begin{aligned}
\hat{\rho}(ts)(u) &= (2\pi)^{-1}\int \exp(-iu\cdot y)\rho(ts)(y)dy \\
&= (2\pi)^{-1}\int \exp(-iu\cdot y)(1/\sqrt{ts})1_{[0,(t,s)]}(y)dy \\
&= (2\pi\sqrt{ts})^{-1}\int_0^t \exp(-iu_1y_1)dy_1\int_0^s \exp(-iu_2y_2)dy_2 \\
&= (2\pi\sqrt{ts})^{-1}(-iu_2)^{-1}(\exp(-itu_1)-1)(-iu_2)^{-1}(\exp(-isu_2)-1)
\end{aligned}$$

Thus,

$$\begin{aligned}
|\hat{\rho}(ts)(u)| &\leq (2\pi\sqrt{ts})^{-1}|u_1u_2|^{-1}|\exp(-itu_1)-1|\,|\exp(-isu_2)-1| \\
&\leq (2\pi\sqrt{ts})^{-1}|u_1u_2|^{-1}|tu_1|\,|su_2| = (ts)/(2\pi\sqrt{ts}),
\end{aligned}$$

which gives that $|\hat{\rho}(ts)(s_1) \dots \hat{\rho}(ts)(s_n)|^2 \leq (ts/(4\pi^2))^n$. Also,

$\int_{\mathbb{R}^{2n}}(1+|s|^2)^{-\frac{(2n+1)}{2}} \leq 2\sigma_{2n}$. The result now follows from lemma 2.2.1(a)

<u>proof of (a).</u> By lemma 2.2.1(b)

$$\left\|\sum_{n=1}^{\infty}(n!2^n)^{-1}H_n(x/\sqrt{2ts})H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\right\|^2_{(L^2)^-} \leq \sum_{n=1}^{\infty}(n!2^{3n-1})^{-1}H_n^2(x/\sqrt{2ts})(ts/\pi^2)^n\sigma_{2n}$$

Now, from [19,formula (8.91.10)] we have

$$\sup_{\substack{ts>0\\ x\in\mathbb{R}}}\left|(\sqrt{\pi}\,n!2^n)^{-\frac{1}{2}}\exp(-x^2/(4ts)H_n(x/\sqrt{2ts})\right| = o(n^{-\frac{1}{12}})$$

Thus for all t,s > 0, $x \in \mathbb{R}$

$$\left\|(2\pi ts)^{-\frac{1}{2}}\exp(-x^2/2ts)\sum_{n=1}^{\infty}(n!2^n)^{-1}H_n(x/\sqrt{2ts})H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\right\|^2_{(L^2)^-}$$

$$\leq \beta/(ts)\exp(-x^2/(2ts))\sum_{n=1}^{\infty}n^{-\frac{1}{6}}(ts/(2\pi^2))^n\sigma_{2n}$$

where β is a constant independent of t and x. Since $\sigma_n = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ ([10, p1427]), this last series is convergent.

<u>proof of (b).</u>

Since $F \in \mathscr{S}^*$, and $\mathscr{S}$ is a nuclear space, there exists a norm on $\mathscr{S}^*$ such that F is continuous with respect to that norm [3]. Thus

$$\left\|\sum_{n=0}^{\infty}(2\pi ts)^{-\frac{1}{2}}(n!2^n)^{-1}H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\int_{-\infty}^{\infty}\exp(-x^2/(2ts)H_n(x/\sqrt{2ts})F(x)dx\right\|^2_{(L^2)^-}$$

$$\leq (2ts\sqrt{\pi})^{-\frac{1}{2}}\exp(-x^2/2ts)C\sum_{n=0}^{\infty}\left[(n!2^n)^{-1}\sup_{\substack{ts>0\\ x\in\mathbb{R}}}\left|(\sqrt{\pi}n!2^n)^{-\frac{1}{2}}\exp(-x^2/4ts)H_n(x/\sqrt{2ts})\right|^2\right.$$

$$\left.\times\,\|H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\|^2_{(L^2)^-}\right]$$

where C is a constant which depends only on F. Now the proof is as in (a).

<u>proof of (c)</u>

$S\Big[F(W(t,s))\Big](\xi) =$

$$S\Big[\sum_{n=0}^{\infty}\Big[\int_{-\infty}^{\infty}(\sqrt{2\pi ts}\,n!2^n)^{-1}\exp(-x^2/2ts)H_n(x/\sqrt{2ts})F(x)dx\Big]H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2})\Big](\xi)$$

$$= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}(\sqrt{2\pi ts}\,n!2^n)^{-1}\exp(-x^2/2ts)H_n(x/\sqrt{2ts})F(x)dx(2/ts)^{\frac{n}{2}}\Big[\Big[\int_{]0,(t,s)]}\xi(u)du\Big]^n\Big]$$

since $S(H_n(\langle\cdot,\rho(ts)\rangle/\sqrt{2}))(\xi) = (2/ts)^{\frac{n}{2}}\Big[\int_{]0,(t,s)]}\xi(\mathbf{u})d\mathbf{u}\Big]^n$, see theorem 1.1.3.

$$= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(2\pi ts)^{-\frac{1}{2}}\exp(-x^2/2ts)(ts)^{-n}(ts)^{\frac{n}{2}}(n!2^{\frac{n}{2}})^{-1}H_n(x/\sqrt{2ts})\langle\xi,1_{]0,(t,s)]}\rangle^n F(x)dx.$$

By [5, formula 32.A,p312] this equals

$$\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(2\pi ts)^{-\frac{1}{2}}\exp(-x^2/2ts)(ts)^{-n}H_n(x;ts)\langle\xi,1_{]0,(t,s)]}\rangle^n F(x)dx$$

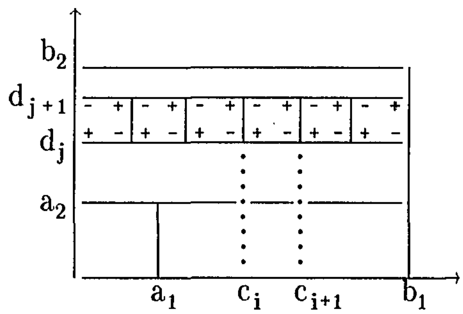
$$= \int_{-\infty}^{\infty}(2\pi ts)^{-\frac{1}{2}}\exp(-x^2/2ts)\Big[\sum_{n=0}^{\infty}(\langle\xi,1_{]0,(t,s)]}\rangle/ts)^n H_n(x;ts)\Big]F(x)dx$$

$$= \int_{-\infty}^{\infty}(2\pi ts)^{-\frac{1}{2}}\exp(-x^2/2ts)\exp(\tfrac{-ts}{2}(\langle\xi,1_{]0,(t,s)]}\rangle/ts)^2 + \langle\xi,1_{]0,(t,s)]}\rangle x/ts)F(x)dx$$

$$= \int_{-\infty}^{\infty}(2\pi ts)^{-\frac{1}{2}}\exp((x^2-2\langle\xi,1_{]0,(t,s)]}\rangle + \langle\xi,1_{]0,(t,s)}]\rangle^2)/2ts)F(x)dx$$

$$= g_{ts} * F\langle\xi,1_{]0,(t,s)]}\rangle.$$

To give some intuitive understanding of the following theorem, recall that for a fixed $s > 0$, $W(s,t)/\sqrt{s}$ is a Brownian motion and formula (1.2.1) is the Ito formula for the process $X(t) = W(s,t)$. Suppose that $F \in \mathscr{S}(R)$, $0 < a_1 < b_1$, and $0 < a_2 < b_2$. Let $P_n = \{c_n\}_{i=1}^{n}$ and $Q_n = \{d_j\}_{j=1}^{n}$ be sequences of partitions of $[0,b_1]$ and $[a_2,b_2]$, respectively, the meshes of both going to 0 as $n \to \infty$.



$$W(b_1,d_{j+1}) - W(b_1,d_j) = \sum_{i=1}^{n} \Delta_{ij} W$$

Using the partition $\{\Delta_{ij} = [c_i,c_{i+1})\times[d_j,d_{j+1})\}_{i,j=1}^{n}$ of $[(a_2,0),(b_1 b_2)]$, we see that

$$\begin{aligned}
\int_{a_2}^{b_2}\int_0^{b_1} F(W(b_1,t))dW(s,t) &= \lim_{n\to\infty} \text{ in qm} \sum_{i,j} F(W(b_1,d_j))\Delta_{ij}W \\
&= \lim_{n\to\infty} \text{ in qm} \sum_j(\sum_i F(W(b_1,d_j))\Delta_{ij}W) \\
&= \lim_{n\to\infty} \text{ in qm} \sum_j F(W(b_1,d_j)[W(b_1,d_{j+1}) - (W(b_1,d_j)] \\
&= \int_{a_2}^{b_2} F(W(b_1,t)dW(b_1,t). \\
&= \int_{a_2}^{b_2} F(W(b_1,t))d_tW(b_1,t).
\end{aligned}$$

So, $\int_{a_2}^{b_2}\int_0^{b_1} F(W(b_1,t))dW(s,t)$ is actually a line integral.

<u>Theorem 2.2.2</u>

Suppose that $F \in \mathscr{S}^*(R)$, $0 < a_1 < b_1$, and $0 < a_2 < b_2$. Then

$$F(W(b_1,b_2)) - F(W(b_1,a_2)) - (F(W(a_1,b_2) - F(W(a_1,a_2)) =$$
$$\int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,r)}F'(W(b_1,r))dsdr - \int_{a_2}^{b_2}\int_0^{a_1} \partial^*_{(s,r)}F'(W(a_1r))dsdr$$
$$+ \frac{1}{2}\int_{a_2}^{b_2}\Big[b_1F''(W(b_1,r)) - a_1F''(W(a_1,r))\Big]dr$$

<u>Remark</u> As the proof will show,

$$F(W(b_1,b_2)) - F(W(b_1,a_2)) = \int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,t)}F'(W(b_1,r)dsdr + \frac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,r)dr$$

For $F \in \mathscr{S}(\mathbb{R})$, by theorem 2.1.1, this gives

$$F(W(b_1,b_2)) - F(W(b_1,a_2)) = \int_{a_2}^{b_2}\int_0^{b_1}F'(W(b_1,r)dW(s,r) + \frac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,r)dr$$
$$= \int_{a_2}^{b_2}F'(W(b_1,r)d_rW(b_1,r) + \frac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,r)dr$$

i.e., the one–dimensional Ito–formula for the process $F(W(b_1,r))$.

<u>proof</u> By theorem 2.2.1

$$S(F(W(b_1,b_2)) - F(W(b_1,a_2)) - (F(W(a_1,b_2)) - F(W(a_1,a_2))(\xi) =$$
$$g_{b_1b_2} * F\langle \xi,1_{]0,(b_1,b_2)]}\rangle - g_{b_1a_2} * F\langle \xi,1_{]0,(b_1,a_2)]}\rangle$$
$$- (g_{a_1b_2} * F\langle \xi,1_{]0,(a_1,b_2)]}\rangle - g_{a_1a_2} * F\langle \xi,1_{]0,(a_1a_2)]}\rangle)$$
$$= \int_{a_2}^{b_2}\frac{d}{dr}(g_{b_1r} * F\langle \xi,1_{]0,(b_1,r)]}\rangle)dr - \int_{a_2}^{b_2}\frac{d}{dr}(g_{a_1r} * F\langle \xi,1_{]0,(a_1,r)]}\rangle)dr$$

It is enough to look at the first summand.

$$\int_{a_2}^{b_2}\frac{d}{dr}(g_{b_1r} * F\langle \xi,1_{]0,(b_1,r)]}\rangle)dr$$
$$= \int_{a_2}^{b_2}\langle \frac{d}{dr}\, g_{b_1r}(\int_0^{b_1}\int_0^r \xi(u,v)dudv - y),F\rangle dr$$

$$= \int_{a_2}^{b_2}\left[\frac{b_1}{2} g''_{b_1 r} * F\langle \xi, 1_{]0,(b_1,r)]}\rangle + \left[\int_0^{b_1}\xi(u,r)du\right] g'_{b_1 r} * F\langle \xi, 1_{]0,(b_1,r)]}\rangle\right] dr$$

$$= \frac{1}{2}\int_{a_2}^{b_2} b_1 g_{b_1 r} * F''\langle \xi, 1_{]0,(b_1,r)]}\rangle dr$$

$$+ \int_{a_2}^{b_2}\left[\left[\int_0^{b_1}\xi(u,r)du\right] g_{b_1 r} * F'\langle \xi, 1_{]0,(b_1,r)]}\rangle\right] dr$$

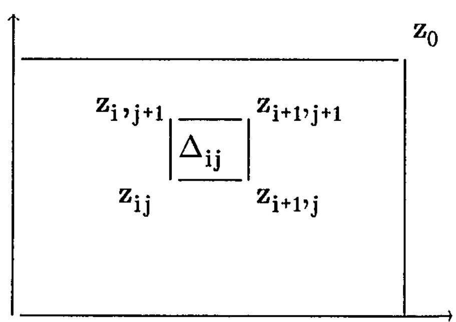
$$= \frac{1}{2}\int_{a_2}^{b_2} b_1 g_{b_1 r} * F''\langle \xi, 1_{]0,(b_1,r)]}\rangle dr + \int_{a_2}^{b_2}\int_0^{b_1} \xi(s,r) g_{b_1 r} * F'\langle \xi, 1_{]0,(b_1,r)]}\rangle dsdr$$

$$= S\left[\frac{1}{2}\int_{a_2}^{b_2} b_1 F''(W(b_1,r))dr + \int_{a_2}^{b_2}\int_0^{b_1} \partial^*_{(s,r)} F'(W(b_1,r))dsdr\right](\xi).$$

The next theorem is a generalization of the Ito formula of Cairoli and Walsh (theorem 1.2.5) which is itself is a generalization of that of Wong and Zakai (theorem 1.2.3). To see this, it is necessary to obtain the relationship between the the integral of a process with respect to the martingale

$$J(\mathbf{z}) = \frac{1}{2}\left[\int_{\mathbb{R}_z \times \mathbb{R}_z}\right] dW(\mathbf{u})dW(\mathbf{v})$$

and the Wong and Zakai integral of some corresponding process. Towards this end, fix z_0 and divide $\mathbb{R}_{z_0}$ into 2^{2n} congruent rectangles Δ_{ij} labeled as shown



Then,

$$J(z_0) = \frac{1}{2}\left[\int_{\mathbb{R}_{z_0}\times\mathbb{R}_{z_0}}\right] dW(\mathbf{u})dW(\mathbf{v}) = \frac{1}{2}\left[\int_{\mathbb{R}_{z_0}\times\mathbb{R}_{z_0}}\right] 1_G(z,z')dW(z)dW(z')$$

$$= \frac{1}{2}\lim_{n\to\infty} \text{in qm} \sum_{\substack{i,j\\k,m}} 1_G(z_{ij},z_{km})\Delta_{ij}W\,\Delta_{km}W$$

$$= \frac{1}{2}\lim_{n\to\infty} \text{in qm}\, 2\sum_{i,j}\left[\sum_{k<i}\Delta_{kj}W \sum_{m<j}\Delta_{im}W\right]$$

$$= \lim_{n\to\infty} \text{in qm} \sum_{i,j}(W(z_{i,j+1}) - W(z_{ij}))(W(z_{i+1,j}) - W(z_{ij})).$$

Set $\Delta_{ij}J_n = (W(z_{i,j+1}) - W(z_{ij}))(W(z_{i+1,j}) - W(z_{ij}))$.

Now, consider

$$\left[\int_{\mathbb{R}_{z_0}\times\mathbb{R}_{z_0}}\right]\varphi(z\vee z')dW(z)dW(z') = \lim_{n\to\infty} \text{in qm} \sum_{\substack{i,j\\k,m}} 1_G(z_{ij},z_{km})\varphi(z_{ij}\vee z_{km})\Delta_{ij}W\,\Delta_{km}W$$

$$= \lim_{n\to\infty} \text{in qm}\, 2\sum_{i,j}\varphi(z_{ij})\left[\sum_{k<i}\Delta_{kj}W\right]\left[\sum_{m<j}\Delta_{im}W\right]$$

$$= 2\lim_{n\to\infty} \text{in qm} \sum_{i,j}\varphi(z_{ij})\Delta_{ij}J_n$$

$$= 2\int_{[\mathbf{0},z_0]}\varphi(z)dJ(z).$$

<u>Theorem 2.2.3</u> For $0 < a_1 < b_1$, $0 < a_2 < b_2$, and $F \in \mathscr{S}^*(R)$

$$F(W(b_1,b_2)) - F(W(b_1,a_2)) - (F(W(a_1,b_2)) - F(W(a_1,a_2)))$$

$$= \frac{b_1}{2}\int_{a_2}^{b_2} F''(W(b_1,t))dt - \frac{a_1}{2}\int_{a_2}^{b_2} F''(W(a_1,t))dt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \overset{*}{\partial}_{(s,t)} F'(W(s,t))dsdt + \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2}\int_0^u \overset{*}{\partial}_{(s,t)}\frac{t}{2} F'''(W(u,t))dsdt\right]du$$

$$+\frac{1}{2}\Big[\Big[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1}-\int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1}+\int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1}-\int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1}$$
$$\Big[\overset{*}{\partial}_{(u,v)}\overset{*}{\partial}_{(s,t)}1_G((s,t),(u,v))F''(W((s,t)\vee(u,v))\Big]dsdtdudv\Big],$$

where G is the set of unordered pairs in $]0,(b_1,b_2)]$.

__Remark__ In view of the previous remarks, for $F \in \mathscr{S}(R)$, one recognizes the last integral in the above formula as $\int_{[(a_1,a_2),(b_1,b_2)]} F''(W(z))dJ(z)$ and we see the Cairoli–Walsh formula.

The conditions that Wong and Zakai place on F insure that F(W(z)) is a martingale on every increasing staircase. In this case only the third and last integrals in the above formula would remain and we also see their formula.

__Lemma 2.2.3__ $\int_{a_2}^{b_2}\int_0^{b_1}\overset{*}{\partial}_{(s,t)}F'(W(b_1,t))dsdt - \int_{a_2}^{b_2}\int_0^{a_1}\overset{*}{\partial}_{(s,t)}F'(W(a_1,t))dsdt =$

$$\int_{a_1}^{b_1}\Big[\int_{a_2}^{b_2}\int_0^{a_1}\overset{*}{\partial}_{(s,t)}\frac{t}{2}F'''(W(u,t))dsdt\Big]du$$

$$+\int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^{t}\overset{*}{\partial}_{(s,t)}\overset{*}{\partial}_{(u,v)}F''(W(u,t))dvdudsdt$$
$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\overset{*}{\partial}_{(s,t)}F'(W(b_1,t))dsdt.$$

__Note__ For $F \in \mathscr{S}(\mathbb{R})$ this reduces to Green's formula (theorem 1.2.4) for $F'(W)$.

__proof__

$$\int_{a_2}^{b_2}\int_0^{b_1}\overset{*}{\partial}_{(s,t)}F'(W(b_1,t))dsdt - \int_{a_2}^{b_2}\int_0^{a_1}\overset{*}{\partial}_{(s,t)}F'(W(a_1,t))dsdt$$
$$=\int_{a_2}^{b_2}\int_0^{a_1}\overset{*}{\partial}_{(s,t)}[F'(W(b_1,t))-F'(W(a_1,t))]dsdt+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\overset{*}{\partial}_{(s,t)}F'(W(b_1,t))dsdt.$$

Now,

$$S\Big[\int_{a_2}^{b_2}\int_0^{a_1}\partial^*_{(s,t)}[F'(W(b_1,t)) - F'(W(a_1,t))]dsdt\Big](\xi)$$

$$= \int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)[g_{b_1t} * F'\langle\xi,1_{]0,(b_1,t)]}\rangle - g_{a_1t} * F'\langle\xi,1_{]0,(a_1,t)]}\rangle]dsdt$$

$$= \int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\Big[\int_{a_1}^{b_1}\frac{d}{du}\, g_{ut} * F'\langle\xi,1_{]0,(u,t)]}\rangle du\Big]dsdt$$

$$= \int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\Big[\int_{a_1}^{b_1}\Big[\frac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle + \Big[\int_0^t\xi(u,v)dv\Big]g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle\Big]du\Big]dsdt$$

$$= \int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\Big[\int_{a_1}^{b_1}\frac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle du\Big]dsdt + \int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\Big[\int_{a_1}^{b_1}\Big[\int_0^t\xi(u,v)dv\Big]g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle du\Big]dsdt$$

$$= \int_{a_1}^{b_1}\Big[\int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\frac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dsdt\Big]du + \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t\xi(s,t)\xi(u,v)g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt.$$

<u>**proof of theorem**</u> By the lemma and theorem 2.2.2,

$$S([F(W(b_1,b_2)) - F(W(b_1,a_2))] - [F(W(a_1,b_2)) - F(W(a_1,a_2))] - \frac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,t))dt + \frac{a_1}{2}\int_{a_2}^{b_2}F''(W(a_1,t))dt)(\xi)$$

$$= \int_{a_1}^{b_1}\Big[\int_{a_2}^{b_2}\int_0^{a_1}\xi(s,t)\frac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dsdt\Big]du + \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t\xi(s,t)\xi(u,v)g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt + \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)g_{b_1t} * F'\langle\xi,1_{]0,(b_1,t)]}\rangle dsdt.$$

Note that

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)g_{b_1t} * F'\langle\xi,1_{]0,(b_1,t)]}\rangle dsdt =$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)\left[g_{st} * F'\langle\xi,1_{]0,(s,t)]}\rangle + \int_s^{b_1}\frac{d}{du}(g_{ut} * F'\langle\xi,1_{]0,(u,t)]}\rangle)du\right]dsdt$$

$$= \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)g_{st} * F'\langle\xi,1_{]0,(s,t)]}\rangle dsdt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)\int_s^{b_1}\tfrac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dudsdt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)\int_s^{b_1}\left[\int_0^t\xi(u,v)dv\right]g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle dudsdt$$

$$- \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)g_{st} * F'\langle\xi,1_{]0,(s,t)]}\rangle dsdt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\xi(s,t)\tfrac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dudsdt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\int_0^t\xi(s,t)\xi(u,v)g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt$$

Thus,

$$S([F(W(b_1,b_2)) - F(W(b_1,a_2))] - [F(W(a_1,b_2)) - F(W(a_1,a_2)]$$

$$- \tfrac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,t))dt + \tfrac{a_1}{2}\int_{a_2}^{b_2}F''(W(a_1,t))dt)(\xi)$$

$$= \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2}\int_0^{a_1}\xi((s,t)\tfrac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dsdt\right]du$$

$$+ \int_{a_2}^{b_2}\int_0^{a_1}\int_{a_1}^{b_1}\int_0^t\xi(s,t)\xi(u,v)g_{ut} * F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt +$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)g_{st} * F'\langle\xi,1_{]0,(s,t)]}\rangle dsdt$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_s^{b_1}\xi(s,t)\tfrac{t}{2}\, g_{ut} * F'''\langle\xi,1_{]0,(u,t)]}\rangle dudsdt$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_{s}^{b_1}\int_{0}^{t}\xi(s,t)\xi(u,v)g_{ut}{}^{*}\,F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt.$$

Observing that we can insert $1_G((s,t),(u,v))$ in the second and fourth integrals above,we see that

$$\int_{a_2}^{b_2}\int_{0}^{a_1}\int_{a_1}^{b_1}\int_{0}^{t}\xi(s,t)\xi(u,v)g_{ut}{}^{*}\,F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt$$
$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_{s}^{b_1}\int_{0}^{t}\xi(s,t)\xi(u,v)g_{ut}{}^{*}\,F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt$$

$$=\int_{0}^{b_2}\int_{0}^{a_1}\int_{a_1}^{b_1}\int_{0}^{t}-\int_{0}^{a_2}\int_{0}^{a_1}\int_{a_1}^{b_1}\int_{0}^{t}+\int_{0}^{b_2}\int_{0}^{b_1}\int_{s}^{b_1}\int_{0}^{t}$$
$$-\int_{0}^{a_2}\int_{0}^{b_1}\int_{s}^{b_1}\int_{0}^{t}+\int_{0}^{a_2}\int_{0}^{a_1}\int_{s}^{b_1}\int_{0}^{t}-\int_{0}^{b_2}\int_{0}^{a_1}\int_{s}^{b_1}\int_{0}^{t}$$
$$\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{ut}{}^{*}\,F''\langle\xi,1_{]0,(u,t)]}\rangle dvdudsdt$$

$$=\int_{0}^{b_2}\int_{0}^{b_1}\int_{s}^{b_1}\int_{0}^{t}-\int_{0}^{b_2}\int_{0}^{a_1}\int_{s}^{b_1}\int_{0}^{t}+\int_{0}^{a_2}\int_{0}^{a_1}\int_{s}^{a_1}\int_{0}^{t}-\int_{0}^{a_2}\int_{0}^{b_1}\int_{s}^{a_1}\int_{0}^{t}$$

$$\left[\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{(s,t)\vee(u,v)}{}^{*}\,F''\langle\xi,1_{]0,((s,t)\vee(u,v))]}\rangle\right]dvdudsdt,$$

which, because of 1_G,

$$=\int_{0}^{b_2}\int_{0}^{b_1}\int_{0}^{b_1}\int_{0}^{t}-\int_{0}^{b_2}\int_{0}^{a_1}\int_{0}^{b_1}\int_{0}^{t}+\int_{0}^{a_2}\int_{0}^{a_1}\int_{0}^{a_1}\int_{0}^{t}-\int_{0}^{a_2}\int_{0}^{b_1}\int_{0}^{a_1}\int_{0}^{t}$$
$$\left[\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{(s,t)\vee(u,v)}{}^{*}\,F''\langle\xi,1_{]0,((s,t)\vee(u,v))]}\rangle\right]dvdudsdt$$

But by symmetry in the variables v and t

$$\int_{0}^{b}\int_{0}^{a}\int_{0}^{a}\int_{0}^{t}\Big[\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{(s,t)\vee(u,v)}{}^{*}$$
$$F''\langle\xi,1_{]0,((s,t)\vee(u,v))]}\rangle\Big]dvdudsdt$$

$$= \frac{1}{2}\int_0^b\int_0^a\int_0^a\int_0^b\Big[\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{(s,t)\vee(u,v)}{}^* F''\langle\xi,1_{]0,((s,t)\vee(u,v))]}\rangle\Big]dvdudsdt$$

Hence,

$$S([F(W(b_1,b_2)) - F(W(b_1,a_2))] - [F(W(a_1,b_2)) - F(W(a_1,a_2)] - \frac{b_1}{2}\int_{a_2}^{b_2}F''(W(b_1,t))dt + \frac{a_1}{2}\int_{a_2}^{b_2}F''(W(a_1,t))dt)(\xi) =$$

$$\int_{a_1}^{b_1}\Big[\int_{a_2}^{b_2}\int_0^u \xi(s,t)\frac{t}{2}\, g_{ut}{}^* \, F'''\langle\xi,1_{]0,(u,t)]}\rangle dsdt\Big]du +$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\xi(s,t)\; g_{st}{}^* \; F'\langle\xi,1_{]0,(s,t)]}\rangle dsdt$$

$$+ \frac{1}{2}\Big[\int_0^{b_2}\int_0^{b_1}\int_0^{b_2}\int_0^{b_1} - \int_0^{b_2}\int_0^{a_1}\int_0^{b_2}\int_0^{a_1} + \int_0^{a_2}\int_0^{a_1}\int_0^{a_2}\int_0^{a_1} - \int_0^{a_2}\int_0^{b_1}\int_0^{a_2}\int_0^{b_1}\Big]$$

$$\Big[\xi(s,t)\xi(u,v)1_G((s,t),(u,v))g_{(s,t)\vee(u,v)}{}^* \, F''\langle\xi,1_{]0,((s,t)\vee(u,v))]}\rangle\Big]dudvdsdt.$$

Bibliography

[1] Cairoli, R., Sur une equation differentielle stochastique, Compte Rendus, Acad. Sc., Paris, 274(June 12,1972), Ser. A, 1739–1742.

[2] Cairloi, R., and Walsh, J.B., Stochastic integrals in the plane, Acta Math., 134(1975), 111–183.

[3] Gelfand, I.M., and Vilenkin, N., Generalized Functions, Vol.4, English translation, Academic Press, New York (1964).

[4] Hida, T., Analysis of Brownian functionals, Carleton Math Lecture Notes, No. 13, Carelton Univ., Ottawa, 1975.

[5] Hida, T., Brownian Motion, Applications of Math., Vol. II, Springer – Verlag, 1980.

[6] Hida, T., Lee ,K., Si,S., Multidimensional parameter white noise and Gaussian random fields, preprint.

[7] Hida, T., Kuo, H.H., Potthoff, J., Streit, L., White Noise: An Infinite Dimensional Calculus, to appear.

[8] Ito, K., Multiple Wiener integrals, J. Math. Soc. Japan, 3(1951), p 157–169.

[9] Ito, K., Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces, Soc. for Industrial and Applied Math., 1984.

[10] Iyanaga, S., and Kawada, Y., editors, Encyclopedic Dictionary of Math., English translation (1977), MIT Press.

[11] Kubo, I., Ito formula for generalized Brownian functionals, Lecture Notes in Control and Information Sci., Vol. 49, Springer – Verlag (1983), 156–166.

[12] Kubo, I., and Takenaka, S., Calculus on Gaussian white noise. III, Proc. Japan Acad. 57A (1981), 433–437.

[13] Kunita, H., and Watanabe, S., On square integrable martingales, Nagoya Math J., Vol. 30(1967), 209–245.

[14] Kuo, H.H., Gaussian Measures in Banach Spaces, Lecture Notes in Math., Vol 463, Springer – Verlag, 1975.

[15] Kuo, H.H.,Brownian functionals and applications, Acta. Math., Vol 1(1983), 175–188.

[16] Kuo, H.H., Donsker's delta function as a generalized Brownian functional and its application, Lecture Notes in Control and Information Sci., Vol. 49(1983), Springer – Verlag, 156–166.

[17] Kuo, H.H., and Russek, A., White noise approach to stochastic integration, 1984, preprint.

[18] Kuo, H.H., Laplacian operators of generalized Brownian functionals, Lecture Notes in Math., Vol 1203, Springer – Verlag (1985), 119–128.

[19] Szego, G., Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., Vol.XXIII(1939).

[20] Wong, E., and Zakai, M., Martingales and stochastic integrals for processes with a multidimensional parameter, Z. Wahrscheinlichkeitstheorie und Verw. Gebeite, 29(1974), 109–122.

[21] Yosida, K., Functional Analysis, 3rd ed., (1971), Springer – Verlag.

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